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MODELING OF IMPACT ON A FLEXIBLE BEAM *

Q.F. Wei , P.S. Krishnaprasad and W.P. Dayawansa
Institute for Systems Research and
Department of Electrical Engineering
University of Maryland, College Park, MD 20742.

Abstract

We consider the problem of modeling dynamical effects of impact of an elastic body on a flexible beam. We derive a nonlinear integral equation by using the Hertz law of impact in conjunction with the beam equation. This equation does not admit a closed form solution. We demonstrate the existence of solutions, derive a reliable numerical method for computing solutions, and compare the numerical results with those obtained by others.

1 Introduction

High precision control of robotic manipulators has become increasingly important in a variety of industrial applications, e.g. laser beam technology, semiconductor safer manufacturing etc. This requires paying extra attention to the usual dynamical effects as well as taking into consideration otherwise ignored features such as dynamical effects due to impact. This paper focuses on the latter aspect.

For the sake of simplicity, we only consider an elastic beam subject to impact forces occuring from contact with an elastic body. Here we restrict attention to the problem of modeling, existence of solutions to the model, and the computational aspects. Issues such as how to control a manipulator to minimize the disturbance effects due to impact will be addressed in the future.

Numerous attempts have been made to accurately model the dynamical effects of impact in robotics-oriented applications in the recent years[1,2,3]. Consideration of displacement and use of Hertz's law of impact at the region of contact seems to be the most successful approach[4]. When the contact involves a flexible beam, Hertz's law of impact leads to a nonlinear integral equation called the Hertz equation, which incorporates the effects of local elastic deformation at the region of contact[5]. This model has been widely applied to various impact situations, and the experimental results obtained in [2][6] well support the validation of this equation. Unfortunately, this nonlinear equation does not admit a closed form solution. Timoshenko[4] used the small-increment method to obtain numerical solutions, and it became the basis for evaluation other approximation methods. Some other approximation methods also give very satisfactory results. One of them is the application of the energy method devised by Zener and Feshbach[8], and applied by Lee[5] to central impact of a sphere on a simply supported beam. These approximation methods were proposed without establishing the convergence or even existence of a unique solution.

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In section 2 of this paper the impact problem is properly formulated and the Hertz equation is derived through the Hertz law of impact. In section 3 we will discuss some basic properties of the Green's function associated with the Euler Bernoulli beam equation. This equation is used to model the motion of the beam. In section 4, we will establish the existence and uniqueness of solutions to the Hertz equation by applying the contraction mapping principle. In section 5, a numerical technique based on the contraction mapping principle is presented. Various examples are discussed, and the results obtained by different approaches are compared.

2 Formulation of the Problem

For our purpose, the impact problem can be formulated as follows; a beam is struck transversely by a spherical mass m with initial moving velocity v_0. We further assume that the Hertz law of impact is valid i.e.,

$$\alpha = K[f(t)]^{2/3}, \tag{2.1}$$

where α is the relative approach of the striking body,

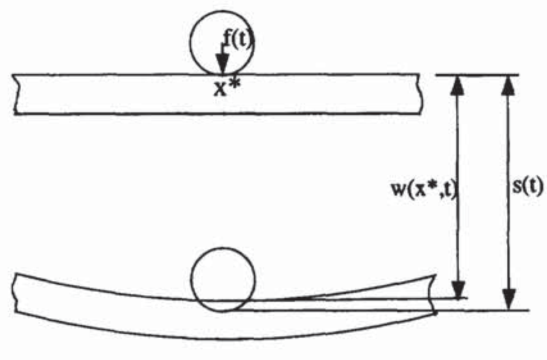

Figure 1: Sketch of the Displacement

$f(t)$ is the contact force and K is the Hertz constant[7], which is determined by the local geometry (i.e. the radius of curvature) and material properties of impacting objects. The relative approach is the difference between the displacement of the beam and the contacting body, measured from the instant of initial contact. Hence,

$$\alpha = s(t) - w(x^*, t), \tag{2.2}$$

where $w(x^*, t)$ is deflection of the beam at the point of contact x^*, $s(t)$ is the displacement of the ball under of the contact force $f(t)$, and is given by,

$$s(t) = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t - \tau)d\tau. \tag{2.3}$$

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¿From the equations (2.3) , (2.1) and (2.2), we obtain the nonlinear integral equation,

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t-\tau)d\tau - w(x^*,t). \quad (2.4)$$

The nonlinear term $K[f(t)]^{2/3}$, precludes any closed form solution for (2.4). Before we can carry out further analysis, it is necessary to represent the deflection of the beam at the point of contact.

3 Deflection of the Beam and Green's Function

If we only restrict attention to transverse vibrations and assume that the beam is long and slender, the transverse shear and torsional effects may be neglected, and the dynamics of the beam can be described by Euler Bernoulli beam equations . The deflection of the beam is in turn obtained by solving the following partial differential equation,

$$\rho\frac{\partial^2 w}{\partial t^2} + EI \bigtriangleup w = \bar{f}(x,t) \quad 0 < x < l, \qquad (3.1)$$

in which $\bigtriangleup = \frac{\partial^4}{\partial x^4}$, l is the length of the beam, ρ is the mass density and EI is the bending stiffness (here ρ and EI are assumed constants), $\bar{f}(x,t)$ is the distributed load. The deflection of the beam is uniquely determined when equation(3.1) is solved under appropriate initial and boundary conditions. If we assume that the beam is at rest just before impact, the initial conditions are,

$$w(x,0) = \dot{w}(x,0) = 0. \qquad (3.2)$$

Various boundary conditions of interest can be described as

$$B_i w(x,t) = 0, \quad i = 1,2, \quad x = 0,l. \qquad (3.3)$$

where B_i is a linear homogeneous differential operator of maximum order 3.

The concentrated load $f(t)$ is obtained as a limiting process of a uniformly distributed load $\bar{f}(x,t)$ over a small range 2δ of the beam. Thus, by letting $\bar{f}(x,t) \to \infty$ while $\delta \to 0$, the contact force $f(t)$ is obtained by,

$$f(t) = \lim_{\delta\to 0, \bar{f}\to\infty} \int_{x^*-\delta}^{x^*+\delta} \bar{f}(x,t)dx \qquad (3.4)$$

With the aid of a Green's function [9], the solution of(3.1) can be expressed as an integral .

Definition 3.1 *A function $G(x,\zeta;t) \in L^2[0,l]$ is called a Green's function of (3.1)-(3.3), if it satisfies the following conditions:*
i) As a function of the argument x, it satisfies the homogeneous differential equation , i.e. $\bar{f} = 0$, everywhere except at $x = \zeta$ where it may have a singularity.
ii) As a function of the argument t, it satisfies the homogeneous differential equation everywhere except at $t = 0$ where it may have a singularity.
iii) As a function of the argument x , $G(x,\zeta;t)$ satisfies the boundary conditions (3.3).
iv) For the initial condition , it satisfies

$$G(x,\zeta;0^+) = 0, \quad and \quad \frac{\partial G(x,\zeta;0^+)}{\partial t} = \delta(x,\zeta)/\rho. (3.5)$$

Note that (i) and (ii) above leads to,

$$\{EI \bigtriangleup + \rho\frac{\partial^2}{\partial t^2}\}G(x,\zeta;t) = \delta(x,\zeta)\delta(t). \qquad (3.6)$$

Since the PDE (3.1)-(3.3) is self-adjoint, $G(x,\zeta;t)$ is symmetric with respect to x and ζ, and can be expressed as an eigenfunction expansion. Therefore, it can be shown that the Green's function for the PDE (3.1)-(3.3) can be expressed in the form (we refer the reader to [10] for details),

$$G(x,\zeta;t)=\sum_{k=1}^{\infty} W_k(x)W_k(\zeta)\frac{sin(w_k t)}{w_k}H(t) \qquad (3.7)$$

where $H(t)$ is the unit step function, $\{W_k(x)\}_{k=1}^{\infty}$ is an orthonormal basis of eigenfunctions and $\{w_k\}_{k=1}^{\infty}$ are the corresponding eigenvalues. It is easy to show that a representation of the solution to the PDE (3.1)-(3.3) in terms of the Green's function $G(x,\zeta;t)$ is

$$w(x,t)=\int_0^t\int_0^l G(x,\zeta;t-\tau)\bar{f}(\zeta,\tau)d\zeta d\tau \qquad (3.8)$$

For the impact problem, since the contact can be treated as point contact, the contact force has the special form (3.4). Hence equation (3.8)can be further simplified as

$$w(x,t) = \int_0^t G(x,x^*;t-\tau)f(\tau)d\tau. \qquad (3.9)$$

For convenience, we write $G(x^*;t)$ instead of $G(x^*,x^*;t)$ in the rest of the paper. From the equations (2.4) and (3.9), the Hertz equation will be

$$\begin{aligned} K[f(t)]^{2/3} &= v_0 t - \frac{1}{m}\int_0^t f(\tau)(t-\tau)d\tau \\ &\quad - \int_0^t G(x^*;t-\tau)f(\tau)d\tau. \quad (3.10) \end{aligned}$$

4 Analysis of the Hertz Equation

Though , to our knowledge, the Hertz equation (3.10) has not been analyzed in detail in the literature, some approximation methods for solving it have been presented in some detail. Our viewpoint is that theoretical analysis is necessary for both validation of this equation and developing efficient numerical methods. The contraction mapping technique is employed here to show that a unique solution exists for the Hertz equation. Before invoking the contraction mapping theorem, some simplifications are necessary. Let,

$$L(t) = t + mG(x^*;t), \qquad \forall t \geq 0. \qquad (4.1)$$

Equation (3.10) can be rewritten as,

$$f(t)^{2/3} = v_0' t - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau, \qquad (4.2)$$

where $v_0' = v_0/K; \qquad m' = mK;$

$$\begin{aligned} f(t) &= [v_0' t - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau]^{3/2} \qquad (4.3) \\ &= [v_0' t - \frac{1}{m'}\int_0^t f(\tau)(t-\tau)d\tau \\ &\quad -\frac{1}{K}\int_0^t f(\tau)G(x^*;t-\tau)d\tau]^{3/2}. \qquad (4.4) \end{aligned}$$

Note that v_0' is assumed to be positive always. Both equations (4.3)and (4.4)will be used in the following analysis.

Theorem 4.1 *(Contraction Mapping Theorem) Let X be a Banach space, and B be a closed subset of X. Let P: B → B be an operator satisfying the following condition:*

$\exists\ \rho < 1$ *such that*

$$||Px - Py|| \leq \rho||x - y||, \qquad \forall x, y \in B.$$

Then
a) P has exactly one fixed point in B (denoted by x^*).
b) For any $x_0 \in B$, the sequence $\{x_n\}_0^\infty$ defined by

$$x_{n+1} = Px_n, \qquad n \geq 0$$

converges to x^*. Moreover,

$$||x_n - x^*|| \leq \frac{\rho^n}{1-\rho}||Px_0 - x_0||.$$

A proof of this well-known theorem can be found in[11]. We will use this theorem as the main tool to show that equation (4.3) has a unique solution by constructing a contraction operator P on an appropriate closed subset B of a Banach space.

Theorem 4.2 *Suppose that the Green's function $G(x^*;t)$ is uniformly bounded over $[0,l]$. Then there exists a small enough $\delta > 0$ such that (4.3) has a unique continuous solution for $t \in [0,\delta]$.*
Proof : *Let $M > 0$ be such that,*

$$|G(x^*;t)| \leq M \quad \forall t \geq 0 \quad and \quad \forall x^* \in [0,l];$$

Let $N > 0$ be a sufficiently large constant. Let $\delta > 0$ be small enough such that,
(i) $\delta \leq [\frac{1}{v_0'+MN/K}]^{1/3}$;
(ii) $\delta \leq \frac{v_0'}{[1/2m'+M/2K]}$;
(iii) $2(\delta^2/m'+M\delta/K)\sqrt{(v_0'+MN/K)\delta} < 1$.
Our Banach space here is $C[0,\delta]$, the space of continuous real valued functions on $[0,\delta]$, endowed with the sup norm, i.e. $||f||_\infty = \sup_{t\in[0,\delta]}|f(t)|$. Let us define the mapping $P: C[0,\delta] \to C[0,\delta]$ by,

$$Pf(t) = [v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau]^{3/2}, \forall t \in [0,\delta] \quad (4.5)$$

The domain of P is defined by,

$$B[0,\delta] = \{f(\cdot) \in C[0,\delta]; N \geq f(\cdot) \geq 0; \quad (4.6)$$
$$v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau \geq 0 \forall t \in [0,\delta]\}.$$

Obviously, $B[0,\delta]$ is a closed subset of the Banach space of continuous functions on $[0,\delta]$.
The rest of the proof is divided into two parts: first, we show that P maps $B[0,\delta]$ into itself. Then we show that P is a contraction mapping on $B[0,\delta]$.
a) *$Pf \geq 0\ \forall f \in B[0,\delta]$ by definition. Let us define mapping $F: B[0,\delta] \to B[0,\delta]$ by,*

$$Ff(t) = [v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau]$$

$f \in B[0,\delta] \Longrightarrow \frac{1}{m'}\int_0^t f(\tau)(t-\tau)d\tau \geq 0$. *Hence,*

$$|Ff(t)| = |[v_0't - \frac{1}{m'}\int_0^t f(\tau)(t-\tau)d\tau$$
$$-\frac{1}{K}\int_0^t f(\tau)G(x^*;t-\tau)d\tau]|$$
$$\leq v_0't + \frac{1}{K}\int_0^t |f(\tau)||G(x^*;t-\tau)|d\tau$$
$$\leq (v_0' + MN/K)t.$$

$Ff(t) \geq 0$ by definition. Therefore,

$$Ff(t) \leq (v_0' + MN/K)t \Longrightarrow$$
$$Pf(t) \leq [(v_0' + MN/K)t]^{3/2} \leq t, \forall t \in [0,\delta].$$

$$\begin{aligned} FPf(t) &= v_0't - \frac{1}{m'}\int_0^t Pf(\tau)(t-\tau)d\tau \\ &\quad -\frac{1}{K}\int_0^t Pf(\tau)G(x^*;t-\tau)d\tau \\ &\geq v_0't - \frac{1}{m'}\int_0^t (t-\tau)\tau d\tau - \frac{1}{K}\int_0^t \tau M d\tau \\ &\geq v_0't - \frac{1}{m'}\frac{t^2}{2} - \frac{t^2M}{2K} \\ &\geq 0 \qquad \forall t \in [0,\delta] \end{aligned}$$

Thus, we have shown that $PB[0,\delta] \subset B[0,\delta]$.
b) $\forall f_1, f_2 \in B[0,\delta]$,

$$\begin{aligned} Pf_1(t) - Pf_2(t) &= [v_0't - \frac{1}{m'}\int_0^t f_1(\tau)L(t-\tau)d\tau]^{3/2} \\ &\quad -[v_0't - \frac{1}{m'}\int_0^t f_2(\tau)L(t-\tau)d\tau]^{3/2} \end{aligned}$$

$$Let \quad x(t) = [v_0't - \frac{1}{m'}\int_0^t f_1(\tau)L(t-\tau)d\tau]^{1/2}$$
$$y(t) = [v_0't - \frac{1}{m'}\int_0^t f_2(\tau)G(t-\tau)d\tau]^{1/2}$$

Since $f_1, f_2 \in B[0,\delta], \Longrightarrow x(t) \geq 0, y(t) \geq 0 \forall t \in [0,\delta]$

$$Pf_1(t) - Pf_2(t) = x^3(t) - y^3(t) \quad \forall t \in [0,\delta]$$
$$|Pf_1(t) - Pf_2(t)| \leq |x^2(t) - y^2(t)||x(t) + y(t)|$$

$$\begin{aligned} x^2(t) - y^2(t) &= v_0't - \frac{1}{m'}\int_0^t f_1(\tau)L(t-\tau)d\tau \\ &\quad -(v_0't - \frac{1}{m'}\int_0^t f_2(\tau)L(t-\tau)d\tau) \\ &= \frac{1}{m'}\int_0^t (f_2(\tau) - f_1(\tau))L(t-\tau)d\tau \end{aligned}$$

$$\begin{aligned} |x^2(t) - y^2(t)| &= |\frac{1}{m'}\int_0^t (f_2(\tau) - f_1(\tau))L(t-\tau)d\tau| \\ &\leq \frac{1}{m'}\int_0^t |f_2(\tau) - f_1(\tau)|(t-\tau)d\tau \\ &\quad +\frac{1}{K}\int_0^t |f_2(\tau) - f_1(\tau)||G(x^*;t-\tau)|d\tau \\ &\leq (t^2/m' + Mt/K)||f_2 - f_1||_\infty \\ &\leq (\delta^2/m' + M\delta/K)||f_2 - f_1||_\infty \end{aligned}$$

$$\begin{aligned}|x(t)+y(t)| &\leq |x(t)|+|y(t)| \\ &\leq 2\sqrt{v_0't+MNt/K} \\ &\leq 2\sqrt{(v_0'+MN/K)\delta}\end{aligned}$$

$$\begin{aligned}|Pf_1(t)-Pf_2(t)| &\leq |x^2(t)-y^2(t)||x(t)+y(t)| \\ &\leq 2(\delta^2/m'+M\delta/K)* \\ &\quad \sqrt{(v_0'+MN/K)\delta}||f_2(\cdot)-f_1(\cdot)||_\infty \\ &\leq \rho||f_2-f_1||_\infty \ \forall t\in[0,\delta]\end{aligned}$$

where $\rho = 2(\delta^2/m'+M\delta/K)\sqrt{(v_0'+MN/K)\delta}$, *and by the property(iii) of* δ, $\rho<1$. *Therefore,*

$$\begin{aligned}||Pf_1-Pf_2||_\infty &= \sup_{t\in[0,\delta]}|Pf_1(t)-Pf_2(t)| \\ &\leq \rho||f_2-f_1||_\infty\end{aligned}$$

so that P *is a contraction mapping on* $B[0,\delta]$.
Finally, using the theorem 4.1, it follows that the mapping P *has a unique fixed point in* $B[0,\delta]$. *It is clear that* f *is a solution of the equation(4.3) over* $[0,\delta]$ *iff* $Pf=f$, *i.e.* f *is a fixed point of* P *over* $B[0,\delta]$. *This completes our proof.*

The above theorem shows that a unique solution exists over $t\in[0,\delta]$ for some small δ. Our interest is to find the impact force variation during the entire contact period. The following theorem will establish this global result.

Theorem 4.3 *Suppose that the equation (4.3) has a local unique solution over* $[0,\delta]$ *for some sufficiently small* δ. *If* $f(\delta)>0$, *then* $\exists\epsilon>0$, *such that the equation(4.3) has a unique solution on* $[0,\delta+\epsilon]$.
Proof : *The argument is similar to the proof of the local version. We will only carry out details of a crucial step here.*
Let $g:[0,\delta]\to R$ *be the unique solution of (4.3) established in the proof of the theorem 4.2. Let* N *be a positive number larger than* $||g||_\infty$. *Let* $\epsilon>0$ *be a small positive constant. Let*

$$D[0,\delta+\epsilon]=\{f\in C[0,\delta+\epsilon]; f|_{[0,\delta]} \quad (4.7)$$
$$v_0't-\frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau\geq 0\forall t\in[0,\delta+\epsilon]\}.$$

Clearly D *is a closed subset of* $(C[0,\delta+\epsilon],||\cdot||_\infty)$.
Let $F: D[0,\delta+\epsilon]\to C[0,\delta+\epsilon]$ *be*

$$Ff(t) = v_0't-\frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau,$$

and, let $P = F^{3/2}$.
We will show that for small enough ϵ, $P(D)\subset D$, *and,* P *is a contraction mapping, thus establishing the theorem. Note that it follows easily as in the proof of theorem 4.2 that*

$$|Pf(t)| \leq N \qquad \forall t\in[0,\delta+\epsilon].$$

The crucial step is to show that $FPf(t)\geq 0 \ \forall t\in[0,\delta+\epsilon]$, $\forall f\in D$. *Now,*
$FPf(t) = f(t) \ \forall t\leq\delta$ *since* $f|_{[0,\delta]}$ *satisfies (4.3). For* $0\leq t\leq\epsilon$,

$$\begin{aligned}FPf(t+\delta) &= v_0'\delta-\frac{1}{m'}\int_0^\delta Pf(\tau)(t+\delta-\tau)d\tau \\ &\quad -\frac{1}{K}\int_0^\delta Pf(\tau)G(x^*;t+\delta-\tau)d\tau \\ &\quad +v_0't-\frac{1}{m'}\int_0^t Pf(\tau+\delta)L(t-\tau)d\tau \\ &= v_0'\delta-\frac{1}{m'}\int_0^\delta Pf(\tau)(\delta-\tau)d\tau \\ &\quad -\frac{1}{K}\int_0^\delta Pf(\tau)G(x^*;\delta-\tau)d\tau \\ &\quad -\frac{t}{m'}\int_0^\delta f(\tau)d\tau-\frac{1}{K}* \\ &\quad \int_0^\delta f(\tau)(G(x^*,t+\delta-\tau)-G(x^*;\delta-\tau))d\tau \\ &\quad +v_0't-\frac{1}{m'}\int_0^t Pf(\tau+\delta)L(t-\tau)d\tau \\ &=[f^{2/3}(\delta)-\frac{t}{m'}\int_0^\delta f(\tau)d\tau-\bar{G}(x^*;t)+\bar{G}(x^*;0) \\ &\quad +v_0't-\frac{1}{m'}\int_0^t Pf(\tau+\delta)L(t-\tau)d\tau]\end{aligned}$$

where, $\bar{G}(x^*;t) = \frac{1}{K}\int_0^\delta f(\tau)G(x^*;t+\delta-\tau)d\tau$. *By the continuity of the Green's function, for small enough* ϵ,

$$|\bar{G}(x^*,t)-\bar{G}(x^*,0)|\leq\frac{1}{2}f^{2/3}(\delta);\forall t\in[0,\epsilon]$$

For such ϵ,

$$FPf(t+\delta) \geq 0 \qquad \forall t\in[0,\epsilon].$$

Hence, we have shown that $PD[0,\delta+\epsilon]\subset D[0,\delta+\epsilon]$. *It is easy to show that* P *is a contraction mapping. Existence of a unique solution on* $[0,\delta+\epsilon]$ *follows at once.*

Remark 4.1 *The global version has the following physical interpretation. The condition* $f(t)>0$ *means that the contact is in progress at* $t\geq 0$; *finally,* $f(T)=0$ *means that the objects are just about to cease to be in contact, i. e.* T *is the impact duration.*

5 Numerical Methods

Some approximation methods to solve the equation (4.3) have been proposed in the literature. One of them is the energy method[5]. It is simple and fast, although less accurate than the small-increment method. Hence, the results obtained by the energy method can be used as a good initial condition for the more accurate methods. The general solution by the energy method has a sinusoidal form:

$$f_0(t) = K_0 sin(\pi t/T_0) \qquad \forall t\in[0,T_0],$$

where, T_0 is the impact duration calculated from the energy method, and K_0 is a constant. Inspired by the contraction mapping theorem, we develop a numerical method using the successive Picard approximations; $f_0, Pf_0, PPf_0, \cdots\cdots$, where the initial condition f_0 is obtained from the energy method, and P is the contraction operator defined by,

$$Pf(t) = [v_0't-\frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau]^{3/2}, \forall t\in[0,T_0].$$

The expression for the first iterate is,

$$f_1(t) = [v_0' t - \frac{1}{m'} \int_0^t f_0(\tau) L(t-\tau) d\tau]^{3/2}, \forall t \in [0, T_0].$$

Numerical results are given below for two examples. In

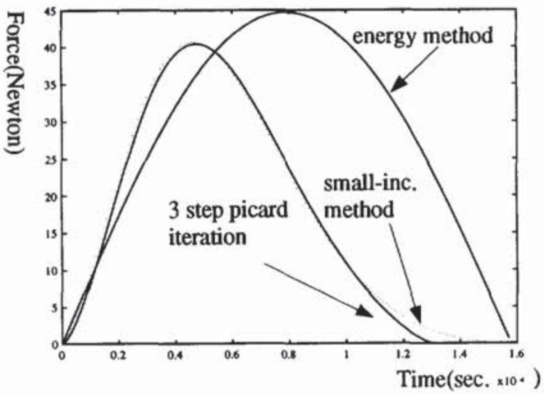

Figure 2: Central impact on a simply supported beam

example 1, impact occurs at the center of a simply supported beam. Fig. 2 shows the solution obtained from the energy method, and gives the force magnitude and duration very close to that obtained by using small-increment method, although the force shape differed in some ways. In order to apply the Picard approximation method to this case, we need to first show that the Green's function associated with this case is uniformly bounded. Details are given in Appendix A. The result obtained by 3-step picard approximation is very good, and the amount of time it takes is just 1/3 of the small-increment method. In example 2, impact is at the tip of a cantilevered beam. We see that fairly large errors occured in using the energy method. On the other hand, the new numerical method gives excellent results even after just one iteration, which has an explicit solution. Fig.3 also shows the fast convergence of this algorithm. Again, the Green's function for this case is uniformly bounded as proven in Appendix B.

6 Conclusion

We have established the existence and uniqueness of solutions of the Hertz equation. A new numerical method is devised based upon the contraction mapping theorem, and various examples have illustrated the usefulness of this method. For simplicity, this paper has dealt solely with the case of a beam. Extensions of the method to multiple space dimensions e.g. plates, are feasible.

Appendix A

For the free vibration, the deflection of a beam is governed by the PDE,

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0, \quad 0 < x < l, \qquad (A.1)$$

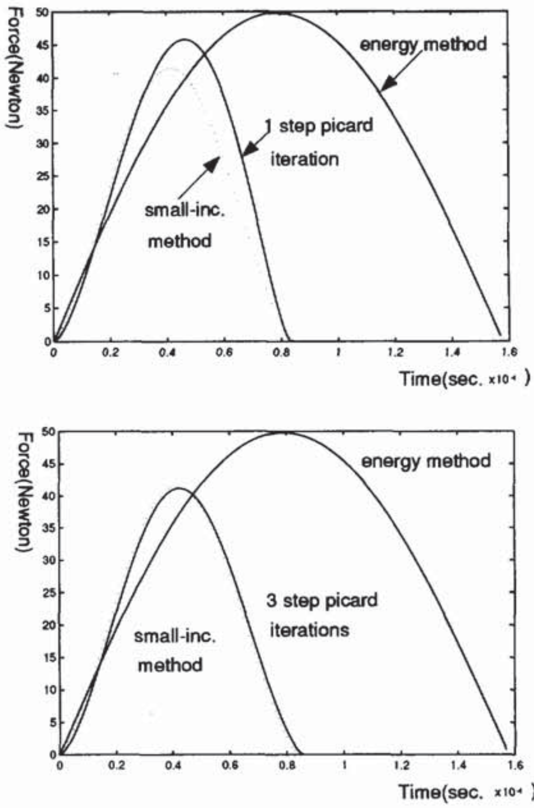

Figure 3: Tip impact on a cantilevered beam

and boundary conditions for the simply supported case are

$$w(0,t) = w''(0,t) = 0, \qquad (A.2)$$
$$w(l,t) = w''(l,t) = 0.$$

The corresponding eigenvalue problem can be written as

$$W'''' = \beta W(x), \qquad (A.3)$$

where β is an eigenvalue and the $W(x) \in L^2[0,l]$ is the corresponding eigenfunction. The boundary conditions are

$$W(0) = W''(0) = 0, \qquad (A.4)$$
$$W(l) = W''(l) = 0.$$

The general solution can be expressed as

$$W(x) = c_1 sin\beta x + c_2 cos\beta x + c_3 sinh\beta x + c_4 cosh\beta x, (A.5)$$

where the c_1 to c_4 are constants to be determined by boundary conditions (A.4). After simplifications, the beam characteristic equation will be

$$sin\beta l = 0 \qquad (A.6)$$

The solutions of equation(A.6) are $\beta_k l = k\pi, k = 1, 2, \cdots$ ....
The normalized eigenfunctions are

$$W_k(x) = \sqrt{2} sin\beta_k x \quad k = 1, 2, \cdots\cdots \qquad (A.7)$$

Lemma A .1 *$\forall x, \zeta \in [0,l]$ and $\forall t > 0$, the Green's function is uniformly bounded.*

Proof: *It is easily checked that (A.3), (A.4) is self - adjoint. Hence, its Green's function can be expressed as*

$$G(x,\zeta;t) = \sum_{k=1}^{\infty} W_k(x)W_k(\zeta)\frac{sinw_k t}{w_k}H(t) \quad (A.8)$$

Here $w_k = \beta_k^2/\sqrt{EI/\rho} = k^2\pi^2/\sqrt{EI/\rho}$.

Now,

$$\begin{aligned} |G(x,\zeta;t)| &= |\sum_{k=1}^{\infty} W_k(x)W_k(\zeta)\frac{sinw_k t}{w_k}H(t)| \\ &\leq \sum_{k=1}^{\infty} |W_k(x)W_k(\zeta)||\frac{sinw_k t}{w_k}H(t)| \\ &\leq 2\sum_{k=1}^{\infty}\frac{1}{w_k} \end{aligned}$$

converges (recall $w_k^2 = \beta_k^4\rho/EI$), $\exists M > 0$, such that $G(x,\zeta;t) \leq M, \forall x, \zeta \in [0,l]$.

Appendix B

The PDE is same as in equation (A.1), and the boundary conditions for the cantilevered beam are

$$\begin{aligned} w(0,t) &= w'(0,t) = 0, \quad (B.1) \\ w(l,t)'' &= w'''(l,t) = 0. \end{aligned}$$

The approach to solve the eigenvalue problem is the same. Again, we write the general solution and plug in the boundary conditions (B.1) into this equation. After simplification, we get the beam characteristic equation,

$$cos\beta l cosh\beta l = -1. \quad (B.2)$$

The orthonormal eigenfunctions are determined by the following equations,

$$\begin{aligned} W_k(x) &= \frac{1}{A_k}\bar{W}_(x), \quad \text{where} \quad (B.3) \\ \bar{W}_k(x) &= \frac{cosh\beta_k x - cos\beta_k x}{cosh\beta_k l + cos\beta_k l} - \frac{sinh\beta_k x - sin\beta_k x}{sinh\beta_k l + sin\beta_k l} \\ \text{and } A_k^2 &= \int_0^l \bar{W}_k^2(x)dx = \frac{cos^2\beta_k l}{sin^4\beta_k l}, k = 1,2,\cdots\cdots\infty. \end{aligned}$$

Lemma B .1 *The orthonormal eigenfuntions $\{W_k(x)\}_{k=1}^{\infty}$ are uniformly bounded.*

Proof: *There are infinitely many solutions to the characteristic equation (B.2), $0 < \beta_1 l < \beta_2 l < \cdots\cdots < \infty$, where $\beta_1 l = 4.73$.*

Note that $cosh\beta_k l > 0$, $cosh\beta_k x > 0$, $sinh\beta_k l > 0$, $sinh\beta_k x > 0$. $\forall x \in [0,l], \forall k$.

$$\begin{aligned} \bar{W}_k(x) &= \frac{cosh\beta_k x - cos\beta_k x}{cosh\beta_k l + cos\beta_k l} - \frac{sinh\beta_k x - sin\beta_k x}{sinh\beta_k l + sin\beta_k l} \\ &= \frac{C_k(x) - D_k(x)}{(cosh\beta_k l + cos\beta_k l)(sinh\beta_k l + sin\beta_k l)} \\ C_k(x) &= (cosh\beta_k x - cos\beta_k x)(sinh\beta_k l + sin\beta_k l); \\ D_k(x) &= (sinh\beta_k x - sin\beta_k x)(cosh\beta_k l + cos\beta_k l). \end{aligned}$$

Now, we carry out the simplifications:

$$\begin{aligned} C_k(x) - D_k(x) &= sinh\beta_k(l-x) + sin\beta_k(x-l) \\ &\quad -cos\beta_k x sinh\beta_k l + cosh\beta_k x sin\beta_k l \\ &\quad +sin\beta_k x cosh\beta_k l - sinh\beta_k x cos\beta_k l \\ |C_k(x) - D_k(x)| &\leq sinh\beta_k(l-x) + |sin\beta_k(x-l)| \\ &\quad +|cos\beta_k x|sinh\beta_k l + cosh\beta_k x|sin\beta_k l| \\ &\quad +|sin\beta_k x|cosh\beta_k l + sinh\beta_k x|cos\beta_k l| \\ &\leq 1/2e^{\beta_k(l-x)} + 1 + e^{\beta_k l} + e^{\beta_k x} \end{aligned}$$

Since $\beta_k l > 4$ $\forall k$, it follows that $sinh\beta_k l > 2.$,

$$\begin{aligned} |\bar{W}_k| &\leq \frac{|C_k(x) - D_k(x)|}{|cosh\beta_k l + cos\beta_k l||sinh\beta_k l + sin\beta_k l|} \\ &\leq \frac{|C_k(x) - D_k(x)|}{(cosh\beta_k l - 1)(sinh\beta_k l - 1)} \\ &\leq \frac{4|C_k(x) - D_k(x)|}{cosh\beta_k l(sinh\beta_k l - 1)}, \end{aligned}$$

It is easy to show that $\frac{1}{A_k} = sin^2\beta_k l cosh\beta_k l$,

$$\begin{aligned} |W_k(x)| &\leq sin^2\beta_k l cosh\beta_k l\frac{4|C_k(x) - D_k(x)|}{cosh\beta_k l sinh\beta_k l} \\ &\leq \frac{8(e^{-\beta_1 x}/2 + e^{\beta_1(x-l)} + e^{-\beta_1 l} + 1)}{1 - e^{-2\beta_1 l}}. \\ &\leq M \quad (B.4) \end{aligned}$$

For the cantilevered beam, the differential operator is also self-adjoint. Thus, the proof that the Green's function is uniformly bounded is similar to lemma A.1.

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