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G–SNAKES: NONHOLONOMIC KINEMATIC CHAINS ON LIE GROUPS

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Abstract

We consider kinematic chains evolving on a finite–dimensional Lie group G under nonholonomic constraints, where snake–like global motion is induced by *shape* variations of the system. In particular, we consider the case when the evolution of the system is restricted to a subspace h of the corresponding Lie algebra $\mathcal{G}$, where h is not a subalgebra of $\mathcal{G}$ and it generates the whole algebra under Lie bracketing. Such systems are referred to as G*–Snakes*. Away from certain singular configurations of the system, the constraints specify a (partial) connection on a principal fiber bundle, which in turn gives rise to a geometric phase under periodic shape variations. This geometric structure can be exploited in order to solve the nonholonomic motion planning problem for such systems.

G–Snakes generalize the concept of nonholonomic Variable Geometry Truss assemblies, which are kinematic chains evolving on the Special Euclidean group $SE(2)$ under nonholonomic constraints imposed by idler wheels. We examine in detail the cases of 3–dimensional groups with real non-abelian Lie algebras such as the Heisenberg group $H(3)$, the Special Orthogonal group $SO(3)$ and the Special Linear group $SL(2)$.

1. Introduction

Of significant interest among mechanical systems subject to nonholonomic constraints are those wherein variations of shape induce, under the influence of the constraints, a global motion of the system. A well–known example is that of a free–floating multibody system in space (e.g. robotic manipulators mounted on orbiting satellites), where periodic movements of the joints induce a reorientation of the system under the nonholonomic constraint of conservation of angular momentum [6,10].

Inspired by the experimental work of Joel Burdick and his students at Caltech [2,3], a novel system that uses the above principle for land locomotion was introduced in [7,8]. There, a Variable Geometry Truss (VGT) assembly consisting of longitudinal repetition of truss modules, each one of which is equipped with idler wheels and linear actuators in a planar parallel manipulator configuration, uses periodic changes of the shape of each module to produce global motion (fig. 1).

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The locomotion principle is not based on direct actuation of wheels, but rather on the nonholonomic constraints imposed on the motion of the system by the rolling–without–slipping of the idler wheels of each module on the supporting plane. This results in a snake–like motion of the VGT assembly, which is not too far, at least in principle, from certain modes of actual snake locomotion [4]. Both the shape and the configuration of the VGT assembly can be described by elements of the Special Euclidean group $SE(2)$, the group of rigid motions on the plane. A system like the VGT assembly constitutes a kinematic chain evolving on this matrix Lie group, with the corresponding velocities given by elements of the Lie algebra of $SE(2)$. Of these velocities, the shape variations can be considered as the controls of the system and they are referred to as *shape controls*. The nonholonomic constraints allow us to express the global motion of the VGT assembly as a function of only the shape and the shape controls and to formulate motion control strategies under periodic shape controls.

This situation can be generalized to kinematic chains evolving on an arbitrary (matrix) Lie group G under a certain class of nonholonomic constraints. In particular, we are interested in groups with a real *non–abelian* Lie algebra $\mathcal{G}$ (of finite dimension n) and ℓ–node kinematic chains evolving on them, subject to ℓ constraints which force the velocities of the system to lie in a subspace of $\mathcal{G}$, which is not a subalgebra of $\mathcal{G}$ but which generates the whole algebra $\mathcal{G}$ under Lie bracketing. We refer to systems of this type as G*–Snakes* and observe that they possess an interesting geometric structure: When $\ell = n$ and the codimension of the constraints is one, the configuration and shape spaces of the system specify a principal fiber bundle [1,11] and the nonholonomic constraints specify a (partial) *connection* on it, at least away from certain configurations which we call nonholonomic singularities (higher codimension cases will be treated elsewhere).

In section 2 of this paper, we consider an ℓ–node kinematic chain evolving on an n–dimensional Lie group. The Wei–Norman representation of G [13], which expresses each element of the group as a product of the one–parameter subgroups of G, and the notion of the adjoint action of G on $\mathcal{G}$ allow us to express in a compact form how the motion of each module of the kinematic chain relates to that of the other modules and to the global motion of the system and how this latter becomes a function of just the shape and the shape controls because of the nonholonomic constraints. We show that the configuration and shape spaces of the G–Snake specify a principal fiber bundle and that the nonholonomic constraints specify a connection on it.

In section 3 we focus on 3–node G–Snakes ($\ell = 3$) evolving on 3–dimensional Lie groups ($n = 3$). In particular, we examine, apart from $SE(2)$, the Heisenberg group $H(3)$, the Special Orthogonal group $SO(3)$ and the Special Linear group $SL(2)$.

In section 4 we comment on possible motion planning schemes for those systems. See also [7,8].

In section 5 we discuss possible further extensions of this work. For reasons having to do with ease of exposition, we limit ourselves to matrix Lie groups in this paper. Extensions to arbitrary Lie groups are easy. For the sake of brevity most proofs are omitted. The interested reader should consult [9].

2. Snakes on Lie Groups

2.1. Preliminaries

Consider a left–invariant dynamical system on a matrix Lie group G with n–dimensional Lie algebra $\mathcal{G}$. For $g \in G$, the *left translation* by g is defined as the map $L_g : G \to G : h \mapsto gh$, for $h \in G$. If e is the identity of G, then T_eL_g is the tangent of the map L_g at e. Consider a curve $g(.) \subset G$. Then, there exists a curve $\xi(.) \in \mathcal{G}$ such that:

$$\dot{g} = T_eL_g \cdot \xi = g\xi \ . \tag{1}$$

Let $\{\mathcal{A}_i,\ i = 1,\ldots,n\}$ be a basis of $\mathcal{G}$ and let $[.,.]$ be the usual Lie bracket on $\mathcal{G}$ defined by $[\mathcal{A}_i,\mathcal{A}_j] = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$. Then, there exist constants $\Gamma^k_{i,j}$, called *structure constants*, such that:

$$[\mathcal{A}_i,\mathcal{A}_j] = \sum_{k=1}^{n} \Gamma^k_{i,j}\mathcal{A}_k \ , \ \text{ for } i,j = 1,\ldots,n. \tag{2}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, i.e. the space of linear functions from $\mathcal{G}$ to $I\!R$. Let $\{\mathcal{A}^\flat_i,\ i = 1,\ldots,n\}$ be the basis of $\mathcal{G}^*$ such that

$$\mathcal{A}^\flat_i(\mathcal{A}_j) = \delta^j_i, \quad \text{for } i,j = 1,\ldots,n, \tag{3}$$

where δ^j_i is the Kronecker symbol. Then the curve $\xi(.) \subset \mathcal{G}$ can be represented as:

$$\xi = \sum_{i=1}^{n} \mathcal{A}^\flat_i(\xi)\mathcal{A}_i \ . \tag{4}$$

Proposition 1 (Wei and Norman [13]) Let $g(0) = e$, the identity of G and let $g(t)$ be the solution of (1). Then, locally around $t = 0$, g is of the form:

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1}e^{\gamma_2(t)\mathcal{A}_2}\cdots e^{\gamma_n(t)\mathcal{A}_n} \ , \tag{5}$$

where the coefficients γ_i are determined (by differentiating (5) and using (1)) by:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \vdots \\ \dot{\gamma}_n \end{pmatrix} = M(\gamma_1,\ldots,\gamma_n) \begin{pmatrix} \mathcal{A}^\flat_1(\xi) \\ \vdots \\ \mathcal{A}^\flat_n(\xi) \end{pmatrix} \ . \tag{6}$$

The matrix M is analytic in γ and depends only on the Lie algebra $\mathcal{G}$ and its structure constants in the given basis. If $\mathcal{G}$ is solvable, then there exists a basis of $\mathcal{G}$ and an ordering of this basis, for which the representation (5) is global. Then the γ_i's can be found by quadratures.

■

Definition 2 (Adjoint Action) For $g \in G$ and $\xi \in \mathcal{G}$, define the *adjoint action* of G on $\mathcal{G}$ denoted $Ad_g : \mathcal{G} \to \mathcal{G}$ by:

$$Ad_g\xi \stackrel{\text{def}}{=} g\xi g^{-1} \ . \tag{7}$$

■

The following definitions are based on [1,11].

Definition 3 (Principal Fiber Bundle) Let S be a differentiable manifold and G a Lie group. A differentiable manifold Q is called a (differentiable) *principal fiber bundle* if the following conditions are satisfied:
1) G acts on Q to the left, freely and differentiably:

$$\Phi : G \times Q \to Q : (g,q) \mapsto g \cdot q \stackrel{\text{def}}{=} \Phi_g \cdot q \ .$$

2) S is the quotient space of Q by the equivalence relation induced by G, i.e. $S = Q/G$ and the canonical projection $\pi : Q \to S$ is differentiable.
3) Q is locally trivial, i.e. every point $s \in S$ has a neighborhood U such that $\pi^{-1}(U) \subset Q$ is isomorphic with $U \times G$, in the sense that $q \in \pi^{-1}(U) \mapsto (\pi(q),\phi(q)) \in U \times G$ is a diffeomorphism such that $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(g \cdot q) = g\phi(q), \forall g \in G$.

For $s \in S$, the *fiber over* s is a closed submanifold of Q which is differentiably isomorphic with G. For any $q \in Q$, the *fiber through* q is the fiber over $s = \pi(q)$. When $Q = S \times G$, then Q is said to be a *trivial* principal fiber bundle.

■

Definition 4 (Connection) Let (Q,S,π,G) be a principal fiber bundle. A *connection* on this principal fiber bundle is a choice of a tangent subspace $H_q \subset T_qQ$ at each point $q \in Q$ (horizontal subspace) such that, if $V_q \stackrel{\text{def}}{=} \{v \in T_qQ | \pi_{*_q}(v) = 0\}$ is the subspace of T_qQ tangent to the fiber through q (vertical subspace), we have:
1) $T_qQ = H_q \oplus V_q$.
2) For every $g \in G$ and $q \in Q$, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.
3) H_q depends differentiably on q.

■

2.2. The ℓ–node G–Snake

We consider a dynamical system that evolves on the Cartesian product $Q = \underbrace{G \times \cdots \times G}_{\ell \text{ times}}$. Its trajectory is a curve $g(.) = (g_1(.),\ldots,g_\ell(.)) \subset Q$. On each copy of G, the system traces a curve $g_i(.) \subset G$, such that

$$\dot{g}_i = T_eL_{g_i} \cdot \xi_i = g_i\xi_i \ , \text{ for } i = 1,\ldots,\ell, \tag{8}$$

where $\xi_i(.) \in \mathcal{G}$. We think of the g_i's as the configuration of the *nodes* of a kinematic structure (in particular, a kinematic chain).

A pair of nodes of the structure constitutes a *module*. The *shape* $g_{i,j}$ of module $\{i,j\}$ corresponding to nodes i and j will be defined as:

$$g_{i,j} = g_i^{-1}g_j = g_{i,i+1}\cdots g_{j-1,j}, \ \ i \le j. \tag{9}$$

Consider the corresponding *shape variations* $\xi_{i,j} \subset \mathcal{G}$ defined by:

$$\dot{g}_{i,j} = T_eL_{g_{i,j}} \cdot \xi_{i,j} = g_{i,j}\xi_{i,j} \ , \ \ i \le j \ . \tag{10}$$

Let the *shape* of the kinematic chain be given by the $(\ell-1)$–tuple $(g_{1,2},g_{2,3},\ldots,g_{\ell-1,\ell}) \in S = \underbrace{G \times \cdots \times G}_{(\ell-1) \text{ times}}$. We call Q the *configuration space* of the kinematic structure and S its *shape space*. We can think of the ξ_i's as characterizing the global motion of the G–Snake system with respect to some global coordinate system, while the $\xi_{i,j}$'s capture the relative motion (or shape variation) of nodes i and j.

From (8), (9) and (10) we get:

$$\xi_i = \xi_{i-1,i} + Ad_{g^{-1}_{i-1,i}}\xi_{i-1}, \ i = 2,\ldots,\ell. \tag{11}$$

Applying (11) iteratively we can express any ξ_i as a function of ξ_1 and of the shape controls $\xi_{1,2}, \ldots, \xi_{i-1,i}$ as follows:

$$\xi_i = \xi_{i-1,i} + Ad_{g^{-1}_{i-1,i}} \xi_{i-2,i-1} + \cdots + Ad_{g^{-1}_{2,i}} \xi_{1,2} + Ad_{g^{-1}_{1,i}} \xi_1 \,. \tag{12}$$

As can be seen from (4), equation (12) is linear in the components of ξ_1 and those of the $\xi_{i,j}$'s. Denote by $\xi^{i,j}$ the vector of components of $\xi_{i,j}$, i.e. $\xi^{i,j} \stackrel{\text{def}}{=} \left(\mathcal{A}_1^\flat(\xi_{i,j}), \ldots, \mathcal{A}_n^\flat(\xi_{i,j})\right)^\top$. Also denote by $\gamma_k^{i,j}$ the coefficients of the Wei–Norman representation corresponding to $g_{i,j} \subset G$.

2.3. Nonholonomic Constraints and Connections on Principal Fiber Bundles

In this section we consider nonholonomic constraints acting on the G–Snake and we show that they specify a connection on the principal fiber bundle associated to our problem.

Codimension 1 Constraint Hypothesis: Assume that the evolution of system (8) on each copy of G is constrained to lie on an $(n-1)$–dimensional subspace h of the Lie algebra $\mathcal{G}$, where h is not a subalgebra of $\mathcal{G}$, i.e. $\xi_i \in h$ for $i = 1, \ldots, \ell$. Then, for some $\mathcal{A}_\kappa^\flat \in \mathcal{G}^*$ (not necessarily an element of the basis $\{\mathcal{A}_i^\flat,\ i = 1, \ldots, n\}$) we have $h = \mathrm{Ker}(\mathcal{A}_\kappa^\flat)$. The constraints $\xi_i \in h$ can then be expressed as:

$$\mathcal{A}_\kappa^\flat(\xi_i) = 0,\ i = 1, \ldots, \ell. \tag{13}$$

The constraints (13) are linear in the components of the global velocity ξ_1 and those of the shape variations $\xi_{i,j}$. This can be made explicit by defining the composite *velocity vector* of the kinematic chain:

$$\begin{aligned}\Xi &\stackrel{\text{def}}{=} \left({\xi^1}^\top\ {\xi^{1,2}}^\top\ \cdots\ {\xi^{\ell-1,\ell}}^\top \right)^\top \\ &= \left(\mathcal{A}_1^\flat(\xi_1)\ \cdots\ \mathcal{A}_n^\flat(\xi_1)\ \mathcal{A}_1^\flat(\xi_{1,2})\ \cdots\ \mathcal{A}_n^\flat(\xi_{\ell-1,\ell})\right)^\top .\end{aligned}$$

Proposition 5 The ℓ nonholonomic constraints (13) can be written in matrix form as:

$$A(g_{1,2}, \ldots, g_{\ell-1,\ell})\,\Xi\, =\, 0\,, \tag{14}$$

where A is a function of only the shape of the system and is a block lower triangular $\ell \times n\ell$ matrix of maximal rank of the form:

$$A = \begin{pmatrix} *_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & \cdots & 0 & 0 \\ *_{1,i} & *_{2,i} & \cdots & *_{i-1,i} & *_{i,i} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \cdots & \ddots & 0 \\ *_{1,\ell} & *_{2,\ell} & \cdots & *_{i-1,\ell} & *_{i,\ell} & \cdots & *_{\ell-1,\ell} & *_{\ell,\ell} \end{pmatrix}, \tag{15}$$

with the $1 \times n$ block $*_{p,q}$, defined for $p \le q$ as:

$$*_{p,q} = \left(\mathcal{A}_\kappa^\flat(Ad_{g^{-1}_{p,q}} \mathcal{A}_1)\ \ \cdots\ \ \mathcal{A}_\kappa^\flat(Ad_{g^{-1}_{p,q}} \mathcal{A}_n) \right).$$

Proof The matrix form (14) is easily derived from (12) and (13). The diagonal blocks $*_{p,p}$ of A have the form $\left(\mathcal{A}_\kappa^\flat(\mathcal{A}_1)\ \cdots\ \mathcal{A}_\kappa^\flat(\mathcal{A}_n)\right)$, therefore they contain at least one non-zero constant term. Thus A has always maximal rank.

∎

Proposition 6 Assume $\ell \ge n$. Partition Ξ as $(\Xi_2\ \Xi_1)$, with Ξ_2 an ℓ–dimensional vector containing the components of ξ_1 (and possibly some components of shape variations), while Ξ_1 is an $(n-1)\ell$–dimensional vector containing *only* components of shape variations. Let the corresponding partition of A be $(A_2\ A_1)$, with A_1 a $(n-1)\ell \times \ell$ matrix and A_2 a locally invertible $\ell \times \ell$ matrix. Then from (14):

$$\Xi_2 = -A_2^{-1}(g_{1,2}, \ldots, g_{\ell-1,\ell}) A_1(g_{1,2}, \ldots, g_{\ell-1,\ell}) \Xi_1\,. \tag{16}$$

Proof Follows from the smooth dependence of A on the shape variables and the maximal rank property of Proposition 5.

∎

Because of equation (16), we call the elements of Ξ_1 the *shape controls.*

The physical significance of this result is that, if the global motion of the G–Snake is characterized by the global motion of its first node (i.e. by ξ_1), then, because of the nonholonomic constraints, variations of the shape controls induce a global motion of the system.

G–Snake configurations where A_2 is singular for all possible choices of the shape controls vector Ξ_1, will be called *nonholonomic singularities.*

Consider now the manifolds Q and S defined in section 2.2 and the canonical projection $\pi : Q \to S$ defined by equation (9), i.e.

$$\pi(g_1, \ldots, g_\ell) \stackrel{\text{def}}{=} (g_1^{-1} g_2, \ldots, g_{\ell-1}^{-1} g_\ell) = (g_{1,2}, \ldots, g_{\ell-1,\ell}). \tag{17}$$

Lemma 7 The quadruple (Q, S, π, G), together with the action Φ of G on Q defined by

$$\begin{aligned}&\Phi : G \times Q \to Q \\ &\quad (g, q) = \left(g, (g_1, \ldots, g_\ell)\right) \mapsto g \cdot q = \left(gg_1, \ldots, gg_\ell\right)\,,\end{aligned} \tag{18}$$

is a trivial principal fiber bundle.

Proof The canonical projection π of equation (17) is differentiable and its differential is

$$\begin{aligned}&\pi_{*_q} : T_q Q \to T_{\pi(q)} S \\ &\quad (g_1 \xi_1, \ldots, g_\ell \xi_\ell) \mapsto (g_{1,2} \xi_{1,2}, \ldots, g_{\ell-1,\ell} \xi_{\ell-1,\ell})\,,\end{aligned} \tag{19}$$

where the $\xi_{i-1,i}$ are given by (11):

$$\xi_{i-1,i} = \xi_i - Ad_{g^{-1}_{i-1,i}} \xi_{i-1},\ i = 2, \ldots, \ell\,. \tag{20}$$

∎

Theorem 8 Away from nonholonomic singularities and when $\ell = n$, i.e. when the number of nonholonomic constraints equals the dimension of the group, the nonholonomic constraints (13) specify a connection on the principal fiber bundle (Q, S, π, G), with the horizontal subspace defined as follows:

$$\begin{aligned}H_q &= \{v \in T_q Q \mid v = (g_1 \xi_1, \ldots, g_\ell \xi_\ell) \text{ and } \xi_i \in h\} \\ &= \{v \in T_q Q \mid v = (g_1 \xi_1, \ldots, g_\ell \xi_\ell) \text{ and } \\ &\qquad \Xi_2 = -A_2^{-1}(\pi(q)) A_1(\pi(q)) \Xi_1 \}\,,\end{aligned} \tag{21}$$

where $\Xi_1 = \left({\xi^{1,2}}^\top {\xi^{2,3}}^\top\ \cdots\ {\xi^{\ell-1,\ell}}^\top\right)^\top$ and $\Xi_2 = \xi^1$.

Proof Due to the left–invariance of our system, $T_qQ = \{v = (g_1\xi_1, \ldots, g_\ell\xi_\ell) \mid \xi_i \in \mathcal{G}\}$. The vertical subspace is (from (18)-(20))

$$\begin{aligned} V_q &= \{v \in T_qQ \mid \pi_{*_q}(\xi) = 0\} \\ &= \{v \in T_qQ \mid (g_{1,2}\xi_{1,2}, \ \ldots, \ g_{\ell-1,\ell}\xi_{\ell-1,\ell}) = 0\} \\ &= \{v \in T_qQ \mid \xi_{1,2} = \cdots = \xi_{\ell-1,\ell} = 0\} \\ &= \{v \in T_qQ \mid \xi_i = Ad_{g_{1,i}^{-1}}\xi_1, \ i = 2, \ldots, \ell\ \} . \end{aligned} \tag{22}$$

Physically, the vertical subspace contains all infinitesimal motions of the kinematic chain that do not alter its shape.

To show property (1) of Definition 4, we first prove that $H_q \cap V_q = \{0\}$ and then that $\dim(T_qQ) = \dim(H_q) + \dim(V_q)$. To show $H_q \cap V_q = \{0\}$, assume that there exists a non–trivial $v = q \cdot \xi \in H_q \cap V_q$. By the definition of V_q, the corresponding shape variations are zero. Thus $\Xi_1 = 0$ and, by the definition of H_q, also $\Xi_2 = 0$. But then $\xi_1 = 0$ and from (22) also $\xi_i = 0$, for $i = 2, \ldots, \ell$. Thus $\xi = 0$. Thus $H_q \cap V_q = \{0\}$. Now observe that, away from the nonholonomic singularities $\dim(H_q) = n\ell - \ell$. Further, $\dim(V_q) = n$. So, when $\ell = n$, $\dim(H_q \oplus V_q) = (n\ell - \ell) + n = (n^2 - n) + n = n^2 = \dim(T_qQ)$. It follows that $H_q \oplus V_q = T_qQ$.

To show property (2), consider $T_q\Phi_g \cdot H_q = g \cdot H_q = g \cdot \{(g_1\xi_1, \ldots, g_\ell\xi_\ell) \mid \xi_i \in h\} \overset{\text{def}}{=} \{(gg_1\xi_1, \ldots, gg_\ell\xi_\ell) \mid \xi_i \in h\}$ and $H_{g\cdot q} = \{v \in T_{g\cdot q}Q \mid v = (g \cdot q) \cdot (\xi_1, \ldots, \xi_\ell)$ and $\xi_i \in h\} = \{(gg_1\xi_1, \ldots, gg_\ell\xi_\ell) \mid \xi_i \in h\}$. Then, obviously, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.

Property (3) is immediate from the smooth dependence of A on the shape and from left–invariance. ∎

3. Three–dimensional Lie Groups

Here we specialize the results of the previous section to kinematic chains on Lie Groups with 3-dimensional real *non-abelian* Lie algebras ($n = 3$). In section 3.1 we consider the Special Euclidean group $SE(2)$, in section 3.2 the Heisenberg group $H(3)$, in section 3.3 the Special Orthogonal group $SO(3)$ and in section 3.4 the Special Linear group $SL(2)$.

We study 3–node, 2–module kinematic chains ($\ell = 3 = n$) on each of these groups by deriving their Wei–Norman representation and by defining the partial connection on the corresponding principal fiber bundle.

Let G be one of the above four matrix Lie groups and $\mathcal{G}$ be the corresponding Lie algebra. Consider the system (8) on G with $g_i \in G$ and $\xi_i \in \mathcal{G}$, for $i = 1, 2, 3$. From Proposition 1, any $g_i \in G$ has a *local* Wei–Norman representation of the form (5). From the system kinematics (equation (9)) we have:

$$g_2 = g_1g_{1,2}, \ g_3 = g_2g_{2,3} = g_1g_{1,2}g_{2,3}, \ g_{1,3} = g_{1,2}g_{2,3}. \tag{23}$$

From (11) we get the corresponding velocity relations. Assume that the evolution of system (8) on each copy of G is constrained to lie on a 2–dimensional subspace h of $\mathcal{G}$, where h is not a subalgebra of $\mathcal{G}$. In [12] the authors present a result showing that, for each of $H(3)$, $SO(3)$ and $SE(2)$, all 2–dimensional subspaces h of $\mathcal{G}$, which are not subalgebras, are equivalent under $\mathrm{Aut}(\mathcal{G})$, the group of automorphisms of $\mathcal{G}$, and can be represented by $h = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\}$ in the basis of $\mathcal{G}$ specified in the following sections. For $SL(2)$, there are 2 classes of such equivalent subspaces that can be represented, respectively, by $h = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\}$ and by $h = \mathrm{sp}\{\mathcal{A}_3, \mathcal{A}_1 + \mathcal{A}_2\}$. Therefore, only nonholonomic constraints corresponding to these subspaces of $\mathcal{G}$ will be considered here.

Define $\Xi_1 = \left(\xi^{1,2\top} \ \xi^{2,3\top}\right)^\top$ and $\Xi_2 = \xi^1$. Observe that, because $\ell = n$, all shape variations appear in the vector Ξ_1, whose choice (as well as the choice of A_1 and A_2) is now unique. Proposition 5 holds with $\Xi = (\Xi_2 \ \Xi_1)$. From Proposition 6 we conclude that the global velocity of the 2–module kinematic chain, as it is characterized by ξ_1, can be expressed as a function of only the shape variables $g_{1,2}, g_{2,3}$ and the shape controls $\xi_{1,2}, \xi_{2,3}$ of the assembly:

$$\Xi_2 = -A_2^{-1}(g_{1,2}, g_{2,3})A_1(g_{1,2}, g_{2,3})\Xi_1 \ . \tag{24}$$

From Theorem 8, equation (24) defines (away from the singularities of A_2) a *connection* on the trivial principal bundle $(S \times G, S, \pi, G)$, with $S = G \times G$.

Our main purpose in this section is to set the stage for a deeper understanding of this novel class of kinematic chains, by cataloguing the low–dimensional possibilities. One case, corresponding to $SE(2)$ has already found a concrete mechanical realization [7,8]. Others might follow, for instance, there are possible connections between $SO(3)$–Snakes and the kinematics of long chain molecules [5].

3.1. $SE(2)$–Snakes

Let $G = SE(2)$ be the Special Euclidean group of rigid motions on the plane and $\mathcal{G} = se(2)$ be the corresponding algebra with the following basis:

$$\mathcal{A}_1=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Then:

$$[\mathcal{A}_1 \ , \ \mathcal{A}_2] = \mathcal{A}_3 \ , \ [\mathcal{A}_1 \ , \ \mathcal{A}_3] = -\mathcal{A}_2 \ , \ [\mathcal{A}_2 \ , \ \mathcal{A}_3] = 0 \ . \tag{26}$$

The algebra $se(2)$ is solvable and, from Proposition 1, any $g \in SE(2)$ has a *global* Wei–Norman representation.

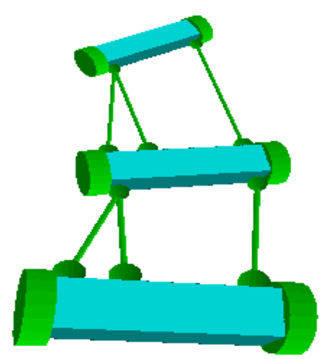

Figure 1: 2–module VGT assembly

From (26) we can see that there are two equivalent 2–dimensional subspaces of $se(2)$ that can generate the whole algebra under Lie bracketing, namely $h_3 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat)$ and $h_2 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_3\} =$

$\mathrm{Ker}(\mathcal{A}_2^\flat)$. Subsequently we will consider only h_2 (which is exactly the case of the system in fig. 1). The nonholonomic constraints $\xi_i \in h_2$ can, then, be expressed as $\mathcal{A}_2^\flat(\xi_i) = 0$, for $i = 1, 2, 3$. Equation (24) holds with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{2,3} & \cos\gamma_1^{2,3} & \sin\gamma_1^{2,3} & 0 & 1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} \\ -\gamma_3^{1,3} & \cos\gamma_1^{1,3} & \sin\gamma_1^{1,3} \end{pmatrix} .$$

See [7,8] for the mechanical interpretation of the nonholonomic singularities of this system. Away from those, equation (24) specifies the connection corresponding to this system.

3.2. $H(3)$–Snakes

Let $G = H(3)$ be the Heisenberg group of real 3×3 upper triangular matrices with diagonal entries equal to one and let $\mathcal{G} = h(3)$ be its algebra with the basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

Then:

$$[\mathcal{A}_1 \ , \ \mathcal{A}_2] = \mathcal{A}_3 \ , \ [\mathcal{A}_1 \ , \ \mathcal{A}_3] = 0 \ , \ [\mathcal{A}_2 \ , \ \mathcal{A}_3] = 0 \ . \tag{28}$$

The algebra $h(3)$ is nilpotent (thus solvable) and, from Proposition 1, any $g \in H(3)$ has a *global* Wei–Norman representation.

From (28) we can see that there is only one possible 2–dimensional subspace of $h(3)$ that can generate the whole algebra under Lie bracketing, namely $h = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat)$. The nonholonomic constraints can, then, be expressed as $\mathcal{A}_3^\flat(\xi_i) = 0$, for $i = 1, 2, 3$. Equation (24) holds with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \gamma_2^{2,3} & -\gamma_1^{2,3} & 1 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ \gamma_2^{1,2} & -\gamma_1^{1,2} & 1 \\ \gamma_2^{1,3} & -\gamma_1^{1,3} & 1 \end{pmatrix} .$$

3.3. $SO(3)$–Snakes

Let $G = SO(3)$ be the Special Orthogonal group of real orthogonal 3×3 matrices with determinant equal to one and let $\mathcal{G} = so(3)$ be the algebra of 3×3 real skew–symmetric matrices, with the following basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

Then:

$$[\mathcal{A}_1 \ , \ \mathcal{A}_2] = \mathcal{A}_3 \ , \ [\mathcal{A}_1 \ , \ \mathcal{A}_3] = -\mathcal{A}_2 \ , \ [\mathcal{A}_2 \ , \ \mathcal{A}_3] = \mathcal{A}_1 \ . \tag{30}$$

The algebra $so(3)$ is simple, thus, the corresponding Wei–Norman representation is only local.

From (30) we can see that there are three equivalent 2–dimensional subspaces of $so(3)$ that can generate the whole algebra under Lie bracketing, namely $h_3 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat)$, $h_2 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_3\} = \mathrm{Ker}(\mathcal{A}_2^\flat)$ and $h_1 = \mathrm{sp}\{\mathcal{A}_2, \mathcal{A}_3\} = \mathrm{Ker}(\mathcal{A}_1^\flat)$. We consider only $h_3 \subset \mathcal{G}$. The nonholonomic constraints $\xi_i \in h_3$ can, then, be expressed as $\mathcal{A}_3^\flat(\xi_i) = 0$, for $i = 1, 2, 3$. Let $\mathrm{c}\gamma \stackrel{\mathrm{def}}{=} \cos\gamma$ and $\mathrm{s}\gamma \stackrel{\mathrm{def}}{=} \sin\gamma$ Equation (24) holds with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathrm{s}\gamma_2^{2,3} & -\mathrm{s}\gamma_1^{2,3}\mathrm{c}\gamma_2^{2,3} & \mathrm{c}\gamma_1^{2,3}\mathrm{c}\gamma_2^{2,3} & 0 & 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathrm{s}\gamma_2^{1,2} & -\mathrm{s}\gamma_1^{1,2}\mathrm{c}\gamma_2^{1,2} & \mathrm{c}\gamma_1^{1,2}\mathrm{c}\gamma_2^{1,2} \\ \mathrm{s}\gamma_2^{1,3} & -\mathrm{s}\gamma_1^{1,3}\mathrm{c}\gamma_2^{1,3} & \mathrm{c}\gamma_1^{1,3}\mathrm{c}\gamma_2^{1,3} \end{pmatrix} .$$

3.4. $SL(2)$–Snakes

Let $G = SL(2)$ be the Special Linear group of real 2×2 matrices with determinant one and let $\mathcal{G} = sl(2)$ be the algebra of real 2×2 matrices of trace zero. Consider the following basis for $sl(2)$:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \mathcal{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \mathcal{A}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{31}$$

Then:

$$[\mathcal{A}_1 \ , \ \mathcal{A}_2] = 2\mathcal{A}_3, \ [\mathcal{A}_1 \ , \ \mathcal{A}_3] = -\mathcal{A}_1, \ [\mathcal{A}_2 \ , \ \mathcal{A}_3] = \mathcal{A}_2. \tag{32}$$

The Wei–Norman representation for this basis is only local. (See however comments in [13] and their Theorem 3. A global representation of $SL(2)$ can be obtained using $\{\mathcal{A}_1, \mathcal{A}_1 - \mathcal{A}_2, \mathcal{A}_3\}$ as a basis of $sl(2)$).

Define $\mathcal{A}_4 \stackrel{\mathrm{def}}{=} \mathcal{A}_1 - \mathcal{A}_2$ and consider the corresponding element of $\mathcal{G}^*$, namely $\mathcal{A}_4^\flat \stackrel{\mathrm{def}}{=} \mathcal{A}_1^\flat - \mathcal{A}_2^\flat$. From (32) we can see that there are two *non*–equivalent 2–dimensional subspaces of $sl(2)$ that can generate the whole algebra under Lie bracketing, namely $h_3 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat)$ and $h_{1,2} = \mathrm{sp}\{\mathcal{A}_3, \mathcal{A}_1 + \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_4^\flat)$.

We will only consider $h_3 \subset \mathcal{G}$. The nonholonomic constraints $\xi_i \in h_3$ can, then, be expressed as $\mathcal{A}_3^\flat(\xi_i) = 0$, for $i = 1, 2, 3$. Equation (24) holds with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2\gamma_2^{2,3} & -2\gamma_1^{2,3}(\gamma_1^{2,3}\gamma_2^{2,3} + 1) & 2\gamma_1^{2,3}\gamma_2^{2,3} + 1 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 2\gamma_2^{1,2} & -2\gamma_1^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1) & 2\gamma_1^{1,2}\gamma_2^{1,2} + 1 \\ 2\gamma_2^{1,3} & -2\gamma_1^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1) & 2\gamma_1^{1,3}\gamma_2^{1,3} + 1 \end{pmatrix}.$$

4. Motion Control

G–Snakes are highly redundant kinematic chains. As a result, there are several possible actuation schemes that implement a desired global motion of the system. In the case of the 2–module $SE(2)$–Snake (2–VGT), we consider a simple such scheme, which relies on one module (e.g. module $\{1, 2\}$) performing the "steering" of the system, while the other (module $\{2, 3\}$) provides the translation mechanism through periodic variations of its shape. The shape controls in (24) are in this case $\Xi_1 = \left(\xi^{1,2\top}\xi^{2,3\top}\right)^\top$. The first three are used for steering and the rest for translating.

Proposition 9 Let the shape of the steering module be fixed, i.e. $\xi_{1,2} = 0$, while the shape of module $\{2,3\}$ changes arbitrarily. Then, the instantaneous motion of the 2–module $SE(2)$–Snake is a rotation around the intersection of the axes of platforms 1 and 2 (c.f. fig. 1). If those axes are parallel, then the assembly instantaneously translates along the common perpendicular to those axes.

■

Consider now a *periodic* variation of the shape controls such that the shape $g_{2,3}$ of module $\{2,3\}$ traces a closed curve in shape–space. It can be shown by numerical integration and by computer simulations [7, 8] that this results in, not just an in–place oscillation of the system, but to a net global motion of the assembly. This motion is in agreement with Proposition 9. If we trace the shape–space curve in reverse, the assembly will move backwards by the same amount. This is associated with the concept of geometric phase of the system, which is discussed in greater detail in [6,10].

Once primitive motions (translations and rotations) can be generated, those can be synthesized, using well known motion planning (i.e. open loop) methods, to solve problems like obstacle avoidance, motion along constrained directions ("car parking" problem) and motion in a confined environment (since the shape of the assembly can be expanded and contracted at will).

Since the geometric picture is exactly the same in the case of $H(3)$, $SO(3)$ or $SL(2)$–Snakes, a similar approach will allow the solution of the motion planning problem also in those cases.

5. Conclusions

In this paper we introduce the concept of *G–Snakes*, which is a class of kinematic chains, with local nonholonomic constraints on each node of the chain, evolving on a Lie group. Shape variations of the system modules induce a snake–like global motion of the system. We provide the framework upon which motion planning strategies based on periodic shape variations can be developed and we offer a catalogue of low–dimensional possibilities. A concrete mechanical realization is associated with $G = SE(2)$. The present framework is applicable not only to nodes arranged in a chain, but to more general tree or ring–like arrangements (thus giving rise to G–Spiders and G–Rings). This may be relevant to the more complicated node arrangement in e.g. molecular structures. Kinematic structures where the Codimension 1 Constraint Hypothesis is relaxed, can also be analyzed using this framework.

Further extensions of this work include the study of geometric phase for each of the groups that we discuss, as well as the study of optimal control problems related to the choice of shape variations that will achieve motion between two desired configurations. Considering the effects of *dynamics* on the system is also a natural extension for this study. We are currently in the process of building a prototype 1–module $SE(2)$–Snake, which uses dynamic effects for locomotion.

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