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SYMPLECTIC MECHANICS AND RATIONAL FUNCTIONS

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Abstract. Certain dynamical systems of many particles have been the subject of intense investigation in Hamiltonian mechanics in recent years. A calculation due to J. Moser [1] has attracted the attention of system theorists since, among other things, it involves a flow on a family of rational functions. Our aim in this paper is to examine the connections between these problems and the geometry of the space of rational functions.

1. Introduction.

Although the exponential-lattice equations [2] describing the evolution of a system of many particles on a line moving under an exponential potential had been known to be completely integrable for some time, it was not until Flaschka [3] discovered a Lax-pair $[L; A]$ for this system that this case became a finite-dimensional analog of the Korteweg-de-Vries equation — the invariants were identified with eigenvalues of the operator L. This involved a change of coordinates and taking into account a basic symmetry of the system, namely translation invariance.

More recently, J. Moser [1] devised yet another change of coordinates for the exponential lattice under special boundary conditions, which involved passing from the pair $[L; A]$ to the rational function $\langle e_n, (\lambda - L)^{-1} e_n \rangle$. That this was possible had a good deal to do with complete integrability as we shall see below.

In what follows, we describe the mechanical systems and their representations briefly and indicate Moser's calculations. We then show

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how these relate to our work with Roger Brockett on the geometry of the space of rational functions. In particular, we will establish what is in some sense, the simplest possible representation of the exponential lattice.

Using the idea of complete integrability (symmetry) we show that Rat (p, q) admits an n-dimensional foliation whose leaves are products of tori and lines. The leaves are level-manifolds in the sense of classical mechanics. It should be possible to apply Kirillov's classification theory [4] to this foliation.

It may be well worth pointing out why system-theorists are interested in flows on rational functions in the first place. The input-output behavior of a linear system of the type

$$\dot{x} = Ax + bu$$

$$y = \langle c, x \rangle$$

is characterized by the rational transfer function $g(\lambda) = \langle c, (\lambda - A)^{-1} b \rangle$. The problem of 'identifying' $g(\lambda)$ from input-output experiments is best formulated as a problem of minimizing a distance function (measuring the quality of fit) on a family of rational functions. It is in this connection that the gradient flow generated by this distance function is of interest and the geometry of the family of rational functions is very relevant to the study of such flows. Other flows appear in dealing with deformations of rational functions, the best-known example being output-feedback.

2. Exponential Lattices: Toda, Flaschka, Moser.

Consider a system of n particles on a line with coordinates $x_1 < \ldots x_k < x_{k-1} < \ldots x_n$ moving freely according to the Hamiltonian,

$$H = \frac{1}{2} \sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{x_k - x_{k+1}} \tag{2.1}$$

where the y_k's are the velocities. We have the system of $2n$ canonical equations

$$\begin{aligned} \dot{x}_k &= \frac{\partial H}{\partial y_k} = y_k \\ \dot{y}_k &= -\frac{\partial H}{\partial x_k} . \end{aligned} \tag{2.2}$$

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The Hamiltonian is invariant under translation $x_k \to x_k+\sigma$ and we have the (linear momentum) conservation

$$\sum_k \dot{y}_k=0.$$

Taking this into account, we pass to system in a $(2n-1)$-dimensional space with coordinates $(a_1 \dots a_{n-1}, b_1 \dots b_n)$ where

$$a_k=\frac{1}{2}\, e^{(x_k-x_{k+1})/2} \qquad k=1, 2, \dots, n-1$$

and

$$b_k=-y_k/2 \quad k=1, 2, \dots, n.$$

If we now arrange these variables in a Jacobi matrix L given by

$$L=\begin{bmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & a_{n-1} & b_n \end{bmatrix}$$

then the system in (a, b)-space is given by the equations

$$\frac{dL}{dt}=[A, L]=AL-LA \tag{2.3}$$

where A is the skew-symmetric matrix associated with L,

$$A=\begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & a_2 & & \\ & -a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & -a_{n-1} & 0 \end{bmatrix}$$

Equation (2.3) is of the Euler-Arnold-Lax form (for the rigid body, perfect fluids and KdeV equation among other systems). The discovery of the Lax pair $[L;A]$ due to Flaschka led to the following. If $U(t)$ denotes the 1-parameter subgroup of $O(n)$ generated by A then the flow on the space of Jacobi matrices is simply the action of $U(t)$ via similarity, i. e. (2.3) is equivalent to the flow

$$L(0) \to U(t)\, L(0)\, U^{-1}(t) = L(t). \tag{2.4}$$

It follows immediately that the spectrum of $L(t)$ is invariant under the flow (2.3). In particular, the first integral H is given by

$$H = 4 \sum_{k=1}^{n-1} a_k^2 + 2 \sum_{k=1}^{n} b_k^2$$

$$= 2\, tr\,(L^2) = 2 \sum_{i=1}^{n} \lambda_i^2$$

where λ_i are the eigenvalues of L. To understand the role of the (invariant) eigenvalues it is useful to recall some properties of L: Since L is Jacobi with $a_i > 0$, e_n is a cyclic vector for L (so is e_1). This implies that L has distinct eigenvalues and the rational function $g(\lambda) = \langle e_n, (\lambda - L)^{-1} e_n \rangle$ is of McMillan degree n. (From elementary realization theory) we have a partial fraction expansion

$$g(\lambda) = \sum_{i=1}^{n} \frac{e^{\alpha_i}}{\lambda - \lambda_i} \tag{2.5}$$

where λ_i are (real) eigenvalues of L and $\alpha_i \in R$, the reals.

As L evolves according to (2.3), the poles of $g(\lambda)$ remain fixed and the residues evolve leaving

$$\sum_{i=1}^{n} e^{\alpha_i} = \langle e_n, e_n \rangle = 1.$$

Moser's main idea in [1], was to recognize that the map

$$L \to \langle e_n, (\lambda - L)^{-1} e_n \rangle$$

is invertible (following the classical moment problem, e. g. Akhiezer [5]) and thus pass from the (a, b) coordinates of Flaschka to the (λ_i, α_i) coordinates where λ_i are invariants. This is very reminiscent of finding action-angle variables [6]. However, to get a complete picture, we carry out some calculations using familiar facts from realization theory.

Note that $g(\lambda)$ has a Laurent expansion,

$$g(\lambda)=\sum_{k=0}^{\infty} h_k/\lambda^{k+1}$$

where

$$h_k=\sum_{i=1}^{n} \lambda_i^k e^{a_i} \quad k=0,1,2,\ldots$$

On the other hand

$$h_k=\langle e_n, L^k e_n\rangle.$$

Therefore

$$\left(\frac{d}{dt}\right) h_k=\langle e_n, [A, L^k]\, e_n\rangle=2\,(h_1 h_k-h_{k+1}) \qquad k=0,1,2,\ldots \tag{2.6}$$

We have an infinite system of ordinary differential equations which leaves invariant the set $h_0=1$. This much was known to Moser. What we shall see now is that this observation together with what is known as scaling in system theory [7] leads us to a particularly simple representation of the Toda lattice. First, note that when L is Jacobi, the rational function $g(\lambda)=\langle e_n, (\lambda-L)^{-1} e_n\rangle$ has Cauchy index n, where we define the Cauchy index to be the winding number,

$$I_{-\infty}^{\infty}(g)= \text{(number of jumps of } g \text{ from } -\infty \text{ to } +\infty) - \text{(number of jumps of } g \text{ from } +\infty \text{ to } -\infty)$$

as λ ranges over the reals from $-\infty$ to $+\infty$.

The Cauchy-index appears in a fundamental way in describing the topology of rational functions. In the next section we summarize the main facts about rational functions.

3. Rational Functions.

In [8], Roger Brockett initiated a program for the study of Rat (n) with the identification problem in mind. The analytic manifold Rat (n) is defined as follows. Consider the set of rational functions of the form $g(\lambda)=q(\lambda)/p(\lambda)$, where $q(\lambda)=q_{n-1}\lambda^{n-1}+\ldots+q_1\lambda+q_0$ and $p(\lambda)=\lambda^n+p_{n-1}\lambda^{n-1}+\ldots+p_1\lambda+p_0$ are relatively-prime polynomials, as an open subspace of R^{2n} as the coefficients (q_i, p_i) vary over the reals.

This subspace together with the analytic manifold structure from R^{2n} is called Rat (n). In [8], Brockett showed that:

$$(a) \quad \text{Rat}\,(n) \text{ splits} \cdot \text{Rat}\,(n) = \bigcup_{p+q=n} \text{Rat}\,(p, q)$$

where on each connected component Rat (p, q) the Cauchy index is constant and takes the value $(p-q)$.

$$(b) \quad \text{Rat}\,(n, 0) \approx \text{Rat}\,(0, n) \approx R^{2n}$$

$$(c) \quad \text{Rat}\,(1, n-1) \approx \text{Rat}\,(n-1, 1) \approx R^{2n-1} \times S^1.$$

Although the geometry of the components is not explicitly understood, some partial results are known. In all this it is good to keep in mind that we have an algebraic map

$$H\colon \text{Rat}\,(n) \to \text{Hank}\,(n)$$

$$g(\lambda) \mapsto H^g = (h_{i+j-2})_{n \times n}$$

where $g(\lambda) = \sum_{k=1}^{\infty} h_k/\lambda^{k+1}$ and H^g is a bilinear form of the Hankel type. That H^g is nondegenerate iff $g(\lambda) \in \text{Rat}\,(n)$ is a result that goes back to Cauchy-Hermite. The Cauchy index is then given by,

$$I_{-\infty}^{\infty}(g) = \sigma(H^g)$$

$$= \text{signature of } H^g.$$

Now any $g(\lambda) \in \text{Rat}\,(n)$ has a realization,

$$g(\lambda) = \langle c, (\lambda - A)^{-1} b \rangle$$

where $c, b \in R^n$ and $A \in L(R^n, R^n)$. The minimality condition

$$\text{spec}\,(A) = \text{poles}\,(g)$$

is equivalent to saying that b and c are cyclic vectors respectively for A and A^*. If we now denote as $\mathfrak{R}$ the collection of all triples $[A, b, c]$ which satisfy this cyclicity condition, then, defining

$$\pi([A, b, c]) = \langle c, (\lambda - A)^{-1} b \rangle$$

$[\mathfrak{R}, \pi, \text{Rat}\,(n)]$ is a principal $Gl_n(R)$ bundle where $Gl_n(R)$ acts on

$\mathfrak{R}$ according to the map:

$$[A, b, c] \to [P^{-1}AP, P^{-1}b, P^{*}c].$$

This is the geometric content of the well-known state-space isomorphism theorem in system theory. However, more is true. The bundle is trivial as it admits a global section,

$$\gamma\colon \text{Rat}(n) \to \mathfrak{R}$$

$$\frac{q(\lambda)}{p(\lambda)} = g(\lambda) \mapsto [A_p, e_n, c_q]$$

where

$A_p =$ unique companion form corresponding to $p(\lambda)$

$c_q = (q_0, q_1, \ldots, q_{n-1})'$

$e_n = (0, 0, \ldots, 0, 1)'$.

Already we see that there is a role for $Gl_n(R)$ as an *internal symmetry group* for linear systems. In a series of papers [7, 9, 10], Brockett and I have worked out a theory of external symmetries for rational functions via certain scalings with physical interpretation. These are:

(1) $g(\lambda) \to g(\alpha\lambda)$; $\alpha > 0$

(frequency scaling)

(2) $g(\lambda) \to g(\lambda + \sigma)$; $\sigma \in (-\infty, \infty)$

(shift of axis)

(3) $g(\lambda) \to mg(\lambda)$; $m > 0$

(magnitude scaling)

(4) $g(\lambda) \to g(\lambda)/(1 + kg(\lambda))$; $k \in (-\infty, \infty)$

(output feedback)

(5) $g(\lambda) = \langle c, (\lambda - A)^{-1} b\rangle \to \langle c, (\lambda - A)^{-1} e^{A\tau} b\rangle$; $\tau \in (-\infty, \infty)$

(time shift).

These scalings act naturally on Rat (n) as one-parameter groups, in the sense that they leave the McMillan degree and Cauchy-index invariant.

Further the scalings 1-5 do not have any fixed points on Rat (n). The idea in [7] was to pick two subsets of scalings generating finite dimensional Lie groups G_A, G_B and examine conditions for these to have nice orbit structures in Rat (p, q). The following general setup is helpful.

Suppose we have a smooth action $\phi: G \times M \rightarrow M$, by a group G on a differentiable manifold M. For every point $m \in M$, there is a map also denoted as $m: G \rightarrow M$

$$m(g) = \phi(g, m).$$

Let dm denote the corresponding derivative map from the tangent-space at the identity e, (Lie algebra $\widetilde{G}$) of G. Then, if dm is of constant rank we have a Lie algebra homomorphism,

$$\widetilde{\phi}: \widetilde{G} \rightarrow U(M) = \text{Lie algebra of smooth vector fields on } M.$$

$$X \rightarrow \widetilde{\phi} X$$

defined by $(\widetilde{\phi} X)(m) = dm(X(e))$.

We can now treat the scalings above as 1-parameter groups acting (freely) on Rat (p, q) and the infinitesimal representations of the scalings are in terms of the vector fields X_α, X_σ, X_m, X_k and X_τ:

$$(1) \quad X_\alpha = -\sum_{i=0}^{n-1} (n-i)\left[q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i} \right]$$

$$(2) \quad X_\sigma = \sum_{i=0}^{n-2} (i+1)\, q_{i+1} \frac{\partial}{\partial q_i} + \sum_{i=0}^{n-2} (i+1)\, p_{i+1} \frac{\partial}{\partial p_i} + n \frac{\partial}{\partial p_{n-1}}$$

$$(3) \quad X_m = \sum_{i=0}^{n-1} q_i \frac{\partial}{\partial q_i}$$

$$(4) \quad X_k = \sum_{i=0}^{n-1} q_i \frac{\partial}{\partial p_i}$$

$$(5) \quad X_\tau = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (A_p^*)_{i+1, j+1}\, q_i \frac{\partial}{\partial p_i}\,.$$

Here A_p is the unique companion form matrix associated with the polynomial $p(\lambda)$ and we are working with local coordinates defined in terms of the coefficients of $p(\lambda)$ and $q(\lambda)$. Suppose we focus on X_τ. First note that using realization theory we can pass from the coordinates

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$(q_0, q_1, \dots, q_{n-1}, p_0, p_1, \dots, p_{n-1})$ to the coordinates $(h_0, h_1, \dots, h_{n-1}, p_0, p_1, \dots, p_{n-1})$ where the h_i are the first n coefficients of the Laurent expansion. The remaining h_i's are of course given by the recursion formula known to Hermite

$$h_{n+j} = -\sum_{i=0}^{n-1} p_i h_{i+j} \qquad j=0,1,2,\dots$$

In (h, p) coordinates,

$$X_\tau = \sum_{i=0}^{n-2} h_{i+1}\frac{\partial}{\partial h_i} - \left(\sum_{i=0}^{n-1} p_i h_i\right)\frac{\partial}{\partial h_{n-1}}$$

Under the shift the Laurent coefficients evolve according to the system

$$(*) \quad \frac{dh_i}{dt} = h_{i+1}, \qquad i=0,1,2,\dots$$

The flow (*) leaves Rat (p,q) invariant. The poles of $g(\lambda) = \sum_{i=0}^{n-1} h_i/\lambda^{i+1}$ are fixed. What is the relationship of (*) to the Toda-Moser system (2.6)? To understand this first recognize that (*) is invariant under the scaling $h_i \to mh_i$, $m>0$. (This is equivalent to $[X_m, X_\tau]=0$). Now in Rat $(n,0)$, we have a representation

$$g(\lambda) = \sum_{i=1}^{n} \frac{e^{\alpha_i}}{\lambda - \lambda_i} \qquad \lambda_i, \alpha_i \in R$$

$\lambda_i \neq \lambda_j$ if $i \neq j$. It follows that $h_0 = \sum_{i=1}^{n} e^{\alpha_i} > 0$ in Rat $(n,0)$.

Now introduce an equivalence relation '$\sim$' in Rat $(n,0)$:

$$g_1(\lambda) \sim g_2(\lambda) \text{ iff } \exists\, m>0 \quad \text{s.t.} \quad g_2 = m\, g_1.$$

The quotient Rat $(n,0)/\sim$ exists, is a differentiable manifold and is diffeomorphic to,

$$\text{Rat}_m(n,0) = \left\{ g(\lambda) = \sum_{i=1}^{n} \frac{e^{\alpha_i}}{\lambda - \lambda_i} : \sum_{i=1}^{n} e^{\alpha_i} = 1 \right\}.$$

Further the triple $[\text{Rat}(n,0), \pi_m, \text{Rat}_m(n,0)]$ is a trivial line bundle where

$$\pi_m\left(\sum_{i=0}^{\infty}\frac{h_i}{\lambda^{i+1}}\right)=\sum_{i=0}^{\infty}\frac{\tilde{h}_i}{\lambda^{i+1}}$$

defined by

$$\tilde{h}_i=h_i/h_0.$$

The vector field X_τ (or the system *) projects down to $\tilde{X}_\tau$, in $\mathrm{Rat}_m(n, 0)$ (recall $[X_\tau, X_m]=0$). The defining equations for $\tilde{X}_\tau$ are:

$$\frac{d\tilde{h}_i}{dt}=\frac{1}{h_0}\frac{dh_i}{dt}-\frac{h_i}{h_0^2}\frac{dh_0}{dt}=\frac{1}{h_0}h_{i+1}-\frac{h_i}{h_0}\cdot\frac{h_1}{h_0}=(\tilde{h}_{i+1}-\tilde{h}_i\,h_1).$$

A time reversal followed by a change of time scale by a factor of 2 brings this to the form (2.6)! Thus Moser's equations for the Toda lattice live naturally on $\mathrm{Rat}_m(n, 0)$ as a projection of the shift.

One last piece of information remains to be recovered. In passing from the (x_k, y_k) coordinates to the 'configuration space' coordinates (a_k, b_k) we have switched to a (moving) coordinate system attached to the center of mass. To recover the center of mass $\sum_{k=1}^{n} x_k=\bar{x}$, first note that,

$$\bar{x}=\sum_{k=1}^{n} y_k=-2\sum_{k=1}^{n} b_k$$

$$=-2tr(L)$$

$$=-2\sum_{i=1}^{n}\lambda_i=\text{constant}.$$

On the other hand under the shift flow on Rat $(n, 0)$,

$$\sum_{i=1}^{n}\frac{e^{\alpha_i}}{\lambda-\lambda_i}\to\sum_{i=1}^{n}\frac{e^{\alpha_i+\lambda_i t}}{\lambda-\lambda_i}$$

and

$$\frac{d}{dt}\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\lambda_i.$$

Thus we also know how to lift an orbit of Moser's equations in $\text{Rat}_m(n, 0)$ to an orbit of the shift in Rat $(n, 0)$. Use the section of the line bundle (Rat $(n, 0)$, π_m, $\text{Rat}_m(n, 0)$) defined by,

$$\gamma : g(\lambda) \mapsto e^{\bar{x}} g(\lambda) \quad !$$

We have thus completed the diagram,

$$\begin{array}{ccc} (x, y) \in R^{2n} & \xrightarrow{\quad \phi \quad} & \text{Rat}\,(n, 0) \\ \text{(Flaschka)} \quad \psi \downarrow & & \pi_m \downarrow \; \uparrow \gamma \\ L \in \mathcal{M} & \xrightarrow[\text{(Moser)}]{\quad \mu \quad} & \text{Rat}_m\,(n, 0) \end{array}$$

where ψ is the projection $(x, y) \to L \in \mathcal{M}$, the space of Jacobi matrices.

$$\mu(L) = \langle e_n, (\lambda - L)^{-1} e_n \rangle$$

$$\phi(x, y) = \langle e_n, (\lambda - L)^{-1} e_n \rangle \cdot \exp(\bar{x}).$$

ϕ and μ are diffeomorphisms, and μ^{-1} is nothing but the Cauer-realization of network theory. ϕ takes orbits of the Toda lattice into orbits of the shift. It is further clear from the equations of the shift,

$$\dot{\alpha}_i = \lambda_i = \frac{\partial H}{\partial \lambda_i}$$

$$\dot{\lambda}_i = 0 = -\frac{\partial H}{\partial \alpha_i}$$

where $H = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2$ that we have a global linearization of the exponential lattice. This explains to some extent the negative results of the Fermi-Pasta-Ulam experiments [18]. At the heart of the matter is the complete integrability of the exponential lattice. This symmetry property is intimately tied up with the geometry of the phase-space. It is both a conceptual and computational advantage to pass from the local-coordinate descriptions of Hamiltonian systems to the symplectic manifold-viewpoint and work with the calculus of differential forms. We proceed to do so in the next section.

4. Scaling and Mechanics.

The geometric formulation of classical mechanics has reached a level where the study of mechanical systems for the large part is the study of the geometry of symplectic manifolds [11]. Symplectic manifolds are the correct generalization of the classical phase-space.

A symplectic manifold is a pair (M, ω) where

(a) M is a smooth manifold

(b) ω is a real, closed, nondegenerate 2-form.

From the nondegeneracy requirement, it follows that $\dim(M) = $ even $= 2n$ say. We say that ω defines a symplectic structure on M. A vector-field $X \in \mathcal{V}(M)$ is said to preserve the symplectic structure if the Lie derivative,

$$D_X \omega = 0.$$

Recall that, in general for $\omega \in \Omega^k(M)$ a k-form,

$$D_X \omega = \lim_{t \to 0} \frac{(\exp. t X)^* \omega - \omega}{t}$$

where $\exp. t X$ is the local 1-parameter group generated by X and $(\exp. t X)^* \omega$ is the pull-back form of ω. For the purposes of calculations the following equality is useful:

$$D_X \omega = \underline{X|} \, d\omega + d\,(\underline{X|}\, \omega)$$

where 'd' denotes the exterior differentiation operator and the contraction operator

$$X| : \Omega^k(M) \to \Omega^{k-1}(M)$$

$$\omega \to \underline{X|}\, \omega$$

is defined by

$$(X| \,\omega)(X_2, X_3, \ldots, X_k) = \omega(X, X_2, \ldots, X_k).$$

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In the present setup, $d\omega=0$ and

$$D_X\,\omega=0 \Leftrightarrow d\,(\underline{X}\rfloor\,\omega)=0$$

or the 1-form $\underline{X}\rfloor\,\omega$ is closed.

Now the map

$$\boldsymbol{\omega}:\ TM \to T^*M$$

$$(x,\xi) \to (x, \omega_x(\xi))$$

is a vector-bundle isomorphism and induces a pairing of sections (vectorfields) of TM with sections (1-forms) of T^*M. We denote this pairing also as

$$\boldsymbol{\omega}:\ \mathcal{U}(M) \to \Omega^1(M)$$

$$X \to \boldsymbol{\omega}(X)=\underline{X}\rfloor\,\omega.$$

If $\theta\in\Omega^1(M)$ is closed we see that $D_{\omega^{-1}(\theta)}\,\omega=0$ and we call the vectorfield $\boldsymbol{\omega}^{-1}(\theta)$ a locally Hamiltonian vectorfield.

If $\theta=dH$ is an exact 1-form where H is a smooth function on M then we say that $X_H=\boldsymbol{\omega}^{-1}(dH)$ is a globally Hamiltonian vectorfield and H is known as the generating function or Hamiltonian of X_H. With this setup, a *Hamiltonian system* is simply a triple (M,ω,X) where (M,ω) is a symplectic manifold and $X\in\mathcal{U}(M)$ such that $D_X\,\omega=0$. If M is connected, simply connected then every such X has a generating function H. The standard example is

$$M=R^{2n} \text{ with coordinates}$$

$(q_1, q_2, \ldots, q_n, p_1, \ldots, p_n)$ and the canonical symplectic structure

$$\omega=\sum_{i=1}^{n} dp_i\wedge dq_i$$

and every Hamiltonian vector field is of the form

$$X_H=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right)$$

where $H=H(q,p)$ is a smooth function on R^{2n}.

Many of the Hamiltonian systems we deal with will have generating functions. In that case, it is immediate that

$$D_{X_H} H = dH(X_H) = 0$$

i. e. the function H is constant along orbits of X. We call H a conserved quantity or first integral. The question of whether there are other conserved quantities (conservation laws) turns out to be one of the most important in mechanics and is a question about the symmetries of a system. Our aim in what follows is to answer this question in relation to the Toda lattice and other such systems.

First note that there are several Lie algebras at hand.

(*a*) $\mathscr{A}_0(M)=$ Lie algebra of locally Hamiltonian vector fields.

(*b*) $\mathscr{A}(M)=$ Lie algebra of globally Hamiltonian vector fields $\mathscr{A}(M) \subset \mathscr{A}_0(M)$ and further it is an easy exercise to verify that

$$[\mathscr{A}_0(M), \mathscr{A}_0(M)] = \mathscr{A}(M) \quad !$$

(see e. g. Simms [12]).

(*c*) The space $C^\infty(M)$ of smooth functions on M can be given the structure of a Lie algebra in the following way. Consider the map

$$P: C^\infty(M) \rightarrow \mathscr{A}(M)$$

$$\phi \rightarrow X_\phi = \omega^{-1}(d\phi)$$

Define the *Poisson bracket* of functions as

$$\{\phi, \psi\} = D_{X_\phi} \psi = X_\phi \psi$$

$$= 2\omega(X_\phi, X_\psi).$$

This bracket satisfies the Jacobi identity (a consequence of $d\omega = 0$!) and with this structure $C^\infty(M)$ is a Lie algebra of functions on M. The map P is a Lie algebra homomorphism.

Given a Hamiltonian system (M, ω, X) we say that a vector field $Y \in \mathscr{A}(M)$ is an *infinitesimal symmetry* of the system if

$$[Y, X] = 0.$$

We say that $\phi \in C^\infty(M)$ is an *integral* of the system if $P(\phi) = X_\phi$ is a symmetry of the system. Further, two such integrals ϕ_1 and ϕ_2 are in *involution* if $\{\phi_1, \phi_2\}=0$. In particular if $X=X_H$ has a generating function H, we will be interested in symmetries $Y \in \mathcal{A}(M)$ for which

$$D_Y H = YH = 0.$$

Let us examine the Toda lattice/shift from this point of view.

Consider Rat $(n, 0)$ with local coordinates (α_i, λ_i) where $g(\lambda)= = \sum_{i=1}^{n} \frac{e^{\alpha_i}}{\lambda-\lambda_i}$. The 2-form $\omega = \sum_{i=1}^{n} d\alpha_i \wedge d\lambda_i$ defines a symplectic structure on Rat $(n, 0)$ and the shift vectorfield,

$$X_\tau = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \alpha_i}$$

which implies that X_τ is globally Hamiltonian on (Rat $(n, 0), \omega$) and has the Hamiltonian $H = \sum_{i=1}^{n} \frac{\lambda_i^2}{2}$. The functions λ_i, $i=1, 2, \ldots, n$ are constant on orbits of X_τ and define symmetries of H by the map

$$\lambda_i \rightarrow X_{\lambda_i} = \lambda_i \frac{\partial}{\partial \alpha_i}.$$

It follows that $[X_{\lambda_i}, X_{\lambda_j}]=0$, so the integrals λ_i, $i=1, 2, \ldots, n$ are in involution.

Now given any globally Hamiltonian system (M, ω, X_H) we say that a system of integrals $H_1, H_2, \ldots, H_k$ of H is *essentially independent* [13], if the set (of singularities)

$$S=\{x \in M \mid \text{rank}\,(dH_{1|x}, \ldots, dH_{k|x}) < k\}$$

has no interior points. Further we measure the *degree of symmetry* of the Hamiltonian H by the number

$$\delta(H) = \max(k)$$

such that there is an essentially independent system of integrals in involution.

Since H is always a candidate for such a system and each of the vector fields X_{H_i} is tangential to the level set,

$$P_c = \{x \in M \mid H_i(x) = c_i, \quad i=1, 2, \ldots, k\}$$

we have the bounds $1 \leqq \delta(H) \leqq n$.

in the case of the shift the set S of singularities is empty and the system $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a complete system of symmetries/integrals for H ($\mathcal{S}(H)=n$). By the implicit function theorem each

$$P_c = \{ g(\lambda) = \sum_{i=1}^{n} \frac{e^{\alpha_i}}{\lambda - \lambda_i} \in \text{Rat}(n, 0) \mid \lambda_i = c_i, \quad i = 1, 2, \ldots, n \}$$

is a smooth submanifold. The vectorfields X_{λ_i} are *complete* and if none of the $c_i = 0$, they act *transitively* on the level manifold P_c. The action may be viewed as an action of R^n

$$R^n \times P_c \to P_c$$

$$\left((t_1, t_2, \ldots, t_n), \sum_{i=1}^{n} \frac{e^{\alpha_i}}{\lambda - c_i} \right), \mapsto \sum_{i=1}^{n} \frac{e^{\alpha_i + c_i t_i}}{\lambda - c_i}.$$

This action is free (i. e. without fixed points) and thus we see that as the c_i vary we obtain a fibration of Rat $(n, 0)$ by level manifolds $\approx R^n$. Notice that if one of the $c_i = 0$, the corresponding α_i remains constant and we pass to a $(2n-2)$ dimension/symplectic manifold to which ω restricts. This reduction of phase-space does not appear to follow directly from the Moser-Flaschka calculations, and should admit physical interpretation. The diffeomorphism ϕ^{-1} in the commutative diagram of Section 3 carries the integrals and level manifolds of the shift to the integrals and level manifolds of the Toda lattice. Finally the level manifolds P_c are all Lagrangian, i. e. the restriction $\omega/P_c = 0$. This is a consequence of the fact that the tangent space to P_c is spanned by commuting vectors at each point of P_c.

One of the features missing from our discussions is the case of compact level manifolds which plays an important role (quasi-periodic motions etc.) in classical mechanics. However a version of this shows up in the periodic Toda lattice (see Flaschka [14], Byrnes [15]).

The question now arises as to how one might extend these ideas to other connected components Rat (p, q). To start with one might consider the shift acting on rational functions $g(\lambda) \in$ Rat (p, q) of the form

$$g(\lambda) = \sum_{i=1}^{\nu} \frac{e^{\alpha_i}}{\lambda - \lambda_i} - \sum_{i=\nu+1}^{n} \frac{e^{\alpha_i}}{\lambda - \lambda_i}$$

where $\nu = p - q > 0$, $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ are all real and the α_i are real.

Once again the shift is Hamiltonian,

$$X_\tau = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \alpha_i}$$

with $H = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2$. However it is not clear what the symplectic structure should be since the possibility of repeated poles for $g(\lambda) \in \mathrm{Rat}(p, q)$, $p - q \neq n$, suggests that (λ_i, α_i) do not give rise to a symplectic atlas — i. e. a covering of coordinates in which the symplectic structure has the simplest form. This is somewhat unsatisfactory.

On the other hand, consider the inclusion

$$i: \mathrm{Rat}(n) \to R^{2n}$$

$$\frac{q(\lambda)}{p(\lambda)} \mapsto (q_0, q_1, \dots, q_{n-1}, p_0, \dots, p_{n-1}).$$

The canonical symplectic structure $\omega = \sum_{i=0}^{n-1} dp_i \wedge dq_i$ pulls-back to $i^* \omega$, since

$$d(i^* \omega) = i^* (d\omega) = 0$$

and nondegeneracy is preserved. Thus $i^* \omega$ (and $-i^* \omega$) defines a symplectic structure. Denote $-i^* \omega = \Omega$. Consider $(\mathrm{Rat}(p, q), \Omega)$ where Ω is restricted to a component and the Hamiltonian vectorfield

$$X_{\tilde{H}} = \sum_{i=0}^{n-1} q_i \frac{\partial}{\partial p_i}$$

where $\tilde{H} = \frac{1}{2} \sum_{i=0}^{n-1} q_i^2 \in C^\infty(\mathrm{Rat}(p, q))$. $X_{\tilde{H}}$ leaves $\mathrm{Rat}(p, q)$ invariant and it integrates to give the flow

$$\frac{q(\lambda)}{p(\lambda)} \to \frac{q(\lambda)}{p(\lambda) + tq(\lambda)}$$

which is simply output-feedback!

The Hamiltonian $\tilde{H}$ is completely symmetric as the system of coefficient functions $\tilde{H}_i = q_i$, $i = 0, 1, 2, \dots, n-1$, form an essentially independent system of integrals of H in involution. However the associated vectorfields $X_{q_i} = q_i \frac{\partial}{\partial p_i}$ are not complete — they eventually enter sets where there is pole-zero cancellation.

The main power of complete symmetry lies in the information it provides on the local geometry of the phase-space/symplectic manifold. This depends on the existence of Abelian actions and as such does not require restriction to symplectic manifolds. We use this point of view to show that Rat (p, q) is fibered via products of circles and lines.

Consider the system of vector fields on Rat (p, q) defined as,

$$X^k = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ([A^* (\tilde{p})]^k)_{i+1, j+1} \, q_j \frac{\partial}{\partial q_i}, \qquad k = 0, 1, 2, \dots, n-1$$

where $A^* (\tilde{p})$ is the adjoint of the unique companion form matrix associated with the polynomial $p(\lambda) = \lambda^n + p_{n-1} \lambda^{n-1} + \dots + p_1 \lambda + p_0$. This system has interesting properties:

Note that X^0 and X^1 are respectively the magnitude scaling and shift vectorfields. Each X^k leaves the poles fixed. X^k generates a flow,

$$(\tilde{q}, \tilde{p}) \to (\exp (t (A^*)^k) \, \tilde{q}, \tilde{p})$$

where

$$\tilde{q} = (q_0, \dots, q_{n-1})' \in R^n$$

$$\tilde{p} = (p_0, \dots, p_{n-1})' \in R^n.$$

It is not very hard to show that

$$(\tilde{q}, \tilde{p}) \in \text{Rat}\,(p, q) \Rightarrow (\exp (t (A^*)^k) \, \tilde{q}, \tilde{p}) \in \text{Rat}\,(p, q).$$

(Use the fact that $\tilde{q}$ is a cyclic vector for A^* iff $\exp (t (A^*)^k) \tilde{q}$ is a cyclic vector for A^* for all t). Thus each X^k is a complete vectorfield on Rat (p, q). Further, since the matrix

$$[\tilde{q}, A^* (\tilde{p}) \tilde{q}, \dots, A^{*n-1} (\tilde{p}) \tilde{q}]$$

is of rank n (Observability!) on Rat (p, q), the tangent vectors $X^0, \dots, X^{(n-1)}$ span an n-dimensional subspace of the tangent space to Rat (p, q) at any point. Finally the vectorfields are in involution i. e.

$$[X^i, X^j] = 0, \qquad i = 0, 1, 2, \dots, n-1.$$

Thus the vectorfields X^k are analogous to a complete system of symmetries and define an Abelian action on Rat (p, q) as follows.

$$\psi: R^n \times \text{Rat}(p, q) \rightarrow \text{Rat}(p, q)$$

$$((t_1, t_2, \ldots, t_n), (\widetilde{q}, \widetilde{p})) \mapsto (\exp(t_1 I + t_2 A^* + \ldots + t_n (A^*)^{n-1}) \widetilde{q}, \widetilde{p})$$

and $A^* = A^*(\widetilde{p})$.

Suppose we denote the 'level manifold',

$$\left\{ \begin{matrix} q(\lambda) \\ p(\lambda) \end{matrix} \in \text{Rat}(p, q) : p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1 \lambda + p_0 \right.$$

where $\widetilde{p} = (p_0, p_1, \ldots, p_{n-1})'$ is fixed$\Big\}$ as $M(\widetilde{p})$. Further let $\mathcal{R}(\widetilde{q}, \widetilde{p})$ denote the reachable set $\bigcup_{t \in R^n} \psi(t, (\widetilde{q}, \widetilde{p}))$.

Then from the properties of the vectorfields X_i noted above it is clear that $X_{i|\mathcal{R}(\widetilde{q}, \widetilde{p})}$ $i = 0, 1, 2, \ldots, n-1$ act transitively on the reachable set $\mathcal{R}(\widetilde{q}, \widetilde{p})$. Further $\mathcal{R}(\widetilde{q}, \widetilde{p}) =$ connected component of $M(\widetilde{p})$. *However $M(\widetilde{p})$ is in general not connected.* It has a finite number of connected components, (see Remark 6 below).

We have,

THEOREM 1. *$\mathcal{R}(\widetilde{q}, \widetilde{p})$ is diffeomorphic to a manifold of the form*

$$T^m \times R^{n-m} = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{m \text{ times}} \times R^{n-m}$$

where T^m is the m torus.

PROOF: The proof of this theorem is essentially the same as the invariant-tori theorem of mechanics (see Arnold [6], Abraham-Marsden [11], Vinogradov-Kupershmidt [13]). We sketch it below.

Let

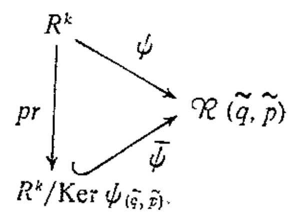
$$\text{Ker}\, \psi_{\widetilde{q}, \widetilde{p}} = \{ t \in R^n \mid \psi(t, (\widetilde{q}, \widetilde{p})) = (\widetilde{q}, \widetilde{p}) \}$$

Ker $\psi_{\widetilde{q}, \widetilde{p}}$ is the isotropy subgroup of the Abelian action ψ. The principal steps in the proof are (*a*) to show that $\operatorname{Ker}\psi_{(\tilde{q},\tilde{p})}$ is a discrete subgroup of R^n

(*b*) there exist m independent vectors h_i in R^n $(0\leq m\leq n)$, such that each

$$\operatorname{Ker}\psi_{(\tilde{q},\tilde{p})}=\{(t_1,\ldots,t_n):(t_1,\ldots,t_n)=\sum_{i=1}^{m} n_i h_i,\ n_i\in Z\}.$$

Thus $\operatorname{Ker}\psi_{(\tilde{q},\tilde{p})}$ is the product of m copies of the infinite cyclic group Z. Since in the diagram



ψ is onto (transitivity!), $\bar{\psi}$ is a diffeomorphism and we have

$$\mathcal{R}(\tilde{q},\tilde{p})\approx R^k/\operatorname{Ker}\psi_{(\tilde{q},\tilde{p})}$$

$$\approx R^k/\underbrace{Z\times Z\times\ldots\times Z}_{m\text{ times}}$$

$$\approx T^m\times R^{n-m}\ \blacktriangleleft$$

REMARK 1: $m=m(\tilde{q},\tilde{p})$ is a integer function of $(\tilde{q},\tilde{p})$. However it is constant on the reachable set $\mathcal{R}(\tilde{q},\tilde{p})$. It is actually constant on an open subset of Rat (p,q). There are two extreme cases possible, $m=0$ and $m=n$. The case $m=0$ uniformly, was encountered in the analysis of Rat $(n,0)$ as the phase-space of the shift/Toda lattice. The case $m=n$ never occurs on Rat (p,q) because then the level manifold (reachable set) $\approx T^n$ would be compact which is impossible.

An example is helpful.

(*a*) $g(\lambda)=\dfrac{q_1\lambda+q_0}{\lambda^2+1}\in$ Rat $(1,1)$

discrete subgroup of R^n

ctors h_i in R^n $(0 \leq m \leq n)$, such

$t_i) = \sum_{i=1}^{m} n_i h_i, \; n_i \in Z\}$.

of the infinite cyclic group Z.

$(\widetilde{q}, \widetilde{p})$

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It is actually constant on an
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then the level manifold
ich is impossible.

$$\psi\left((t_1, t_2), g(\lambda)\right) = \frac{e^{t_1}\left(\bar{q}_1(t_2)\,\lambda + \bar{q}_0(t_2)\right)}{\lambda^2+1}$$

where

$$\begin{pmatrix} \bar{q}_0(t_2) \\ \bar{q}_1(t_2) \end{pmatrix} = \begin{pmatrix} \cos(t_2) & -\sin(t_2) \\ \sin(t_2) & \cos(t_2) \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

Clearly the level manifold $\mathcal{R}(g(\lambda)) \approx S^1 \times R^1$.

$$(b) \quad g(\lambda) = \frac{q_1 \lambda + q_0}{\lambda^2} \in \text{Rat}(1, 1)$$

Here

$$\psi\left((t_1, t_2), g(\lambda)\right) = \frac{e^{t_1}\left(\bar{q}_1(t_2)\,\lambda + \bar{q}_0(t_2)\right)}{\lambda^2}$$

where

$$\begin{pmatrix} \bar{q}_0(t_2) \\ \bar{q}_1(t_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

Here the level manifold $\mathcal{R}(g(\lambda)) \approx R^2$.

REMARK 2: The vectorfields $X^0, X^1, \ldots, X^{n-1}$ define an integrable ($\because [X^i, X^j] = 0$) n-plane distribution and therefore we have proved,

THEOREM 2. *Each connected component* Rat (p, q) *admits an n-dimensional foliation whose leaves are diffeomorphic to $T^m \times R^{n-m}$ where m is constant on an open set. Further on* Rat $(n, 0)$, $m=0$ *and the foliation is actually a (trivial) fibration.*

REMARK 3: Theorem 2 appears to be the correct local version of a long-standing conjecture due to Brockett.

REMARK 4: It is actually possible to obtain an estimate of the maximum value of m on Rat (p, q). If $\sigma = p - q$, then

$$\max(m) = \left[\frac{n - |\sigma|}{2}\right]$$

we leave the proof as an exercise to the reader. Further, the number m satisfies a semicontinuity property illustrated by a slight modification of the previous example. Consider

$$g_\varepsilon(\lambda)=\frac{q_1\lambda+q_0}{\lambda^2+\varepsilon}\in \text{Rat}\,(1,1)$$

$0\leq\varepsilon<\varepsilon_0$. Now the level manifold $\mathcal{R}\,(g_\varepsilon(\lambda))\approx S^1\times R^1$ for $\varepsilon\in(0,\varepsilon_0)$ and we denote $m\,(\varepsilon)=1$, $\varepsilon\in(0,\varepsilon_0)$. But $\mathcal{R}\,(g_0(\lambda))\approx R^2$ and $m\,(0)=1$. Thus $\lim_{\varepsilon\to 0} m\,(\varepsilon)\geq m\,(0)$.

Finally one would like to understand how output feedback behaves with respect to this foliation. In general, if we are given an integrable r-plane distribution τ generating a foliation $\mathcal{F}$ of a manifold M, then the normal bundle $\nu\,(\mathcal{F})=\tau^\perp\,(\mathcal{F})$ of the foliation is the sub-bundle of cotangent bundle T^*M defined by the cotangent vectors which vanish on τ. If a Riemannian metric is given on M, then ν can be identified with the field of $(k-r)$-planes perpendicular to τ where $k=\dim\,(M)$.

On Rat (p, q), if we adopt the Riemannian metric defined by

$$ds^2=\sum_{i=0}^{n-1}(dp_i)^2+(dq_i)^2$$

then the normal-planes to the foliation of Theorem 2 are spanned by the vectors $\dfrac{\partial}{\partial p_i}$, $i=0,1,2,\ldots,n-1$. In particular, output-feedback defined by

$$X=\sum_{i=0}^{n-1}q_i\frac{\partial}{\partial p_i}$$

acts on the normal bundle!

REMARK 5: The normal bundle is extremely important in the study of foliations as it leads to deep topological results [16], [17]. We intend to go into some of these questions in a future paper.

REMARK 6: The connectivity μ of a level manifold $M\,(\tilde{p})$ is determined in the author's paper [25]. There it is shown that

$$\mu\,(M\,(\tilde{p}))\leq 2^{K(\tilde{p})}$$

where $K\,(\tilde{p})=$ the number of distinct real roots of the polynomial $p\,(s)$.

5. Co-adjoint Orbit Theory.

The appearance of symplectic structures in areas of mathematics not quite directly related to mechanics is now better understood as a consequence of Kirillov's work in representation theory [4]. Let G be a Lie group and $\tilde{G}$ it's Lie algebra and $\tilde{G}^*$ the dual of $\tilde{G}$ as a vector space. Thus $l \in \tilde{G}^*$ is a linear functional on $\tilde{G}$. Now G acts on $\tilde{G}^*$ by the well-known co-adjoint action generated by its *infinitesimal version*

$$\tilde{G} \times \tilde{G}^* \to \tilde{G}^*$$

$$(\xi, l) \mapsto ad_\xi\, l$$

when

$$(ad_\xi\, l)(\eta) = l([\xi, \eta]) \text{ for } \xi, \eta \in \tilde{G}.$$

Suppose we denote an orbit of the co-adjoint action as $\mathcal{O}_l$. Then $\mathcal{O}_l$ has the structure of a homogeneous space of G. It is a striking fact that the tangent space $T_l(\mathcal{O}_l)$ carries a nondegenerate, skew-symmetric bilinear form

$$\Omega_l(\xi_1, \xi_2) = l([\bar{\xi_1}, \bar{\xi_2}])$$

where $T_l(\mathcal{O}_l)$ is isomorphic to $\tilde{G}/Z_l$, $Z_l = \{\xi \in \tilde{G} \colon ad_\xi\, l = 0\}$ and $\bar{\xi_1}, \bar{\xi_2}$ are representatives $\in \tilde{G}$ of the equivalence classes (tangent vectors) ξ_1, ξ_2. By translation Ω_l defines the (Kirillov) symplectic structure on $\mathcal{O}_l$ [4]. There are several implications of this construction.

(*a*) All orbits $\mathcal{O}_l \subset \tilde{G}^*$ are even-dimensional.

(*b*) The natural transitive action of G on each $\mathcal{O}_l$ leaves the Kirillov symplectic structure Ω invariant.

(*c*) The vectorfields generating the action of G on $(\mathcal{O}_l, \Omega_l)$ are locally Hamiltonian.

Thus each $(\mathcal{O}_l, \Omega_l)$ is a homogeneous symplectic manifold with $G \supseteq G_s$ the group of symmetries for any Hamiltonian system on $(\mathcal{O}_l, \Omega_l)$. As an example, consider the group of real invertible $n \times n$ matrices $GL_n(R)$ acting via similarity transformations on its Lie algebra

$gl_n(R)=$ all $n\times n$ matrices. This is also the co-adjoint action since gl_n can be identified with gl_n^* using the traceform,

$$gl_n \times gl_n \to R$$

$$(X, Y) \mapsto tr\,(X'\,Y).$$

The (Jordan) orbits are all homogeneous (*wrt* Gl_n) symplectic manifolds of dimension $=n^2-\sum_{k=1}^{t}(2k-1)\,n_k$ where $n_1\geq n_2\geq\ldots\geq n_t$ are the degrees of the nontrivial invariant factors associated with an orbit.

We have a slightly stronger notion [12] of homogeneity: a symplectic manifold (M,ω) is called a *Hamiltonian G-space* for a Lie group G if we have a Lie algebra homomorphism

$$\mu:\ \widetilde{G}\to C_R^{\infty}(M)$$

$$\xi\mapsto\phi_\xi$$

from the Lie algebra of G to the Poisson bracket algebra of (M,ω), satisfying

(*a*) each Hamiltonian vectorfield $P(\phi)=X_{\phi_\xi}$ is complete

(*b*) any two points $m_1, m_2 \in M$ can be joined by an integral curve of $P(\phi_\xi)$ for some $\xi\in\widetilde{G}$.

Every Hamiltonian G-space is a homogeneous symplectic G-manifold. Further the significance of Kirillov's construction follows from the fact that every Hamiltonian G-space is a covering space of an orbit $\mathcal{O}_l$ in $\widetilde{G}^*$. The covering map is

$$\pi:\ (M,\omega)\to(\mathcal{O}_l,\Omega_l)$$

$$m\in M\mapsto\widetilde{l}_m\in\mathcal{O}_l$$

where

$$\widetilde{l}_m(\xi)=\phi_\xi(m).$$

The result is due to Kostant (see [11]).

In this connection, we have two open questions.

1. Is Rat (p, q) with the feedback symplectic structure $\Omega = \sum_{i=0}^{n-1} dq_i \wedge dp_i$ a Hamiltonian G-space?

2. If so, what is the associated co-adjoint orbit?

6. Continuum Limits.

In his classic paper [18], Toda has considered the problem of the limit of the exponential lattice as the number of particles $N \to \infty$. The Gel'fand-Levitan equation plays an important role. Here I would like to indicate briefly how one might attack this problem from the point of view of realization theory.

The weighting pattern $\omega(x)$ of the linear system

$$\frac{dZ}{dx} = A\,Z(x) + b\,u(x)$$
$$y(x) = \langle c, Z(x) \rangle \tag{6.1}$$

is given by,

$$\omega(x) = \langle c, e^{Ax} b \rangle.$$

Here $A \in L(R^n, R^n)$, $b \in R^n$, $c \in R^n$ and b and c are respectively cyclic vectors for A and A^* (minimality). First note that $\omega(x)$ satisfies the differential equation

$$p\left(\frac{\partial}{\partial x}\right)\omega = 0 \tag{6.2}$$

where

$$p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1\lambda + p_0$$

is the characteristic polynomial of A. From minimality ω satisfies no such equation of lower order. The shift acts on the n-dimensional manifold of solutions to (6.2) as the 1-parameter group action,

$$\omega(x) \to \omega(x+t) \qquad t \in R.$$

Denoting $\omega(t+x)$ as $\omega(t, x)$ we have the first order partial differential equation,

$$\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x}$$

If we denote as $g(t, \lambda)$ the Laplace transform

$$g(t, \lambda) \triangleq \int_0^\infty e^{-\lambda x} \omega(t, x) \, dx$$

then the poles of $g(t, \lambda)$ are invariant under the shift. Since the Korteweg-de-Vries (KdeV) equation has an infinite number of conservation laws one might ask if there is any connection between this and shift acting on families of weighting patterns of systems that are infinite-dimensional versions of (*).

For example if $\omega(x)$ is an entire function of the exponential type then it is known [19] that we have always a (bounded) realization

$$\omega(x) = \langle c, e^{Ax} b \rangle$$

where $b, c \in l_2(Z^+)$ the Hilbert space of square summable sequences and $A: l_2(Z^+) \to l_2(Z^+)$ is a bounded operator. This family does rule out many interesting transfer functions. In what follows, we use Lax's method [20] to establish a connection with the KdeV equation.

Let

$$L = L(t)$$

$$= \frac{\partial^2}{\partial x^2} + \alpha \, \omega(t, x)$$

denote the Schrodinger operator with potential. Here α is a constant to be determined. Let $A_0 = \frac{\partial}{\partial \lambda}$.

Then,

$$\frac{\partial L}{\partial t} = [A_0, L] \text{ iff } \frac{\partial}{\partial t} \omega = \frac{\partial}{\partial x} \omega.$$

In this case we say that A_0 is a Lax-pair for the shift. It follows immediately that,

$$U^*(t) \, L(t) \, U(t) = L(0)$$

where $U(t)$ is the 1-parameter group of unitary operators generated by A_0, satisfying.

$$\frac{\partial}{\partial t} U(t) = A_0 U.$$

have poles it is not clear how one might work out a realization theory for them. In any case they do not admit bounded realizations.

7. Final Remarks.

In this paper we have constructed a symplectic equivalence of the Toda lattice and the shift on Rat $(n, 0)$ the space of rational functions of index n. In our efforts to understand the complete-symmetry property of various mechanical systems on Rat (p, q) we have shown that Rat (p, q) admits an n-dimensional foliation whose leaves are products of tori and lines. This is very close to the invariant Tori theorem of classical mechanics. The infinite dimensional analog of the shift leads to elliptic functions.

I would like to acknowledge the original inspiration given by Robert Hermann, David Kazhdan and Roger Brockett in my efforts to understand the connections between system theory and problems in analytical mechanics. In particular, the ideas in [21] and [22] are related to my work here. Professor Brockett deserves special thanks for his encouragement during the preparation of this paper.

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