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On families of systems and deformations

P. S. KRISHNAPRASAD†§ and C. F. MARTIN‡‖

Parameter variations are present in most physical systems. In some cases, such variations can be safely ignored, and one might, for instance, design control loops for some average parameter values. However, in many interesting cases, the variations have to be taken into account in order to design good, or even adequate, control algorithms. Furthermore, the concerns of system reliability demand predictions of possible consequences of large deviations in parameters. Although some of the work in adaptive control is in this spirit, until recently there has not been any systematic effort towards a theory of systems with parameter variations. We argue here that the concept of families of systems is basic to such an effort. Whereas the necessary tools for the study of individual linear systems with fixed parameters are contained in the theory of differential equations and linear algebra, the techniques of Lie theory, differential geometry and algebraic geometry play an essential role in the study of families of systems. The core of this paper is concerned with the geometric characterizations of certain families of systems that appear in control and identification problems. We also isolate some of the ways in which families of systems degenerate as parameter variations become large. For the purposes of exposition, we work mostly with the so-called 'topological case' (over $\mathbb{R}$) as opposed to the algebraic geometric case (over $\mathbb{C}$).

Nomenclature

$\mathbb{R}$	real line : $\mathbb{R}_+$ positive half line
$\mathbb{C}$	complex numbers : $\mathbb{Z}$ integers
$\Sigma_{n,m,p}$	manifold of linear systems (triples $[A, B, C] \simeq \mathbb{R}^{n^2+n(m+p)}$)
$\Sigma_{n,m,p}{}^{r,0}$	manifold of minimal systems $\subseteq \Sigma_{n,m,p}$
$\Sigma_{n,m}{}^{r}$	manifold of completely controllable pairs $[A, B] \subseteq \mathbb{R}^{n^2+nm}$
Λ	parameter space for a family
$Gl(n)$	group of invertible $n \times n$ matrices (over $\mathbb{R}$ unless specified)
$L(n\,;\,m)$	$m \times n$ matrices
$\mathscr{F}$	feedback group
rat (n)	$2n$ dimensional analytic manifold of strictly proper rational functions of degree n (see Brockett 1976)
rat (μ, ν)	connected component of rat (n) of rational fractions of index $(\mu - \nu)$
G_m	stabilizer subgroup at $m \in M$ of a group G acting on a manifold M
$\mathscr{O}_m$	orbit through m

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$\simeq$	symbol for diffeomorphism
$\sigma=(\sigma_1, \ldots, \sigma_n)$	flag partition associated with a system
$\rho=(\rho_1, \ldots, \rho_n)$	Kronecker partition associated with a system
fl (σ)	flag manifold defined by the partition σ
$\mathcal{O}_{A,B}$	orbit of $[A, B]$ under the feedback group
$\mathcal{F}_{A,B}$	stabilizer of $[A, B]$ in $\mathcal{F}$
$\tilde{\mathcal{Q}}_{A,B}$	orbit of $[A, B]$ under Gl (n) action
$\mathcal{V}_{A,B}$	flag–filtration associated with pair $[A, B]$

1. Introduction

Systems with variable parameters are not simply mathematical structures. Tunable electronic circuits constitute a pervasive example of such systems. In the selection of fluid machinery, one takes into account the variation of operating characteristics with the Reynolds number among other factors. Faced with specific design problems (such as broadband matching or flow regulation), the engineer usually approaches these problems with tools familiar in the study of individual systems. This leaves open the question of developing systematic approaches to such problems in different contexts. The concept of a ' family of systems ' appears to be central to any investigation of systems with parameters. Before we formalize this intuitive notion, we consider several instances in which families of systems arise.

Desoer (1977) models the occurrence of ' stray ' elements (such as stray capacitances and lead inductances) in electronic circuits as small parametric effects. In particular, these effects together with the presence of sluggish elements such as chokes, may be represented by models of thef orm (O'Malley 1978)

$$\dot{x}=f(x, y, z, u, t)$$

$$\epsilon\dot{y}=g(x, y, z, u, t)$$

$$\dot{z}=\frac{1}{\mu}\, h(x, y, z, u, t)$$

Here, a small ϵ represents the stray elements and a large μ is associated with the sluggish elements. A perturbation analysis (singular in ϵ and regular $1/\mu$) is necessary to correctly take into account these parametric effects. The point is that one can say quite a bit about such a two-parameter family of systems from the point of view of asymptotics. Closely related in a formal sense are the cheap control problems of linear system theory.

In a quite different setting, we see parametric families of systems arising from scaling operations (Krishnaprasad 1977, Brockett and Krishnaprasad 1980). Let $g(s)=q(s)/p(s)$ denote a proper rational function of degree n, where

$$q(s)=q_{n-1}s^{n-1}+q_{n-2}s^{n-2}+\ldots\ +q_0$$

and

$$p(s)=s^n+p_{n-1}s^{n-1}+\ \ldots\ +p_1s+p_0$$

are fixed. Consider now the associated family

$$F_g=\left\{\frac{mg(\alpha s+\sigma)}{1+mkg(\alpha s+\sigma)}:\ \alpha, m\in\mathbb{R}_+,\quad k, \sigma\in\mathbb{R}\right\}.$$

F_g has the interpretation as a collection of systems (transfer functions) obtained by performing various scaling operations, including frequency scaling (α), magnitude scaling (m), exponential scaling (σ) and output feedback (k). Families of this type are important in identification problems for a number of reasons. For instance, if two experimenters attempt to identify or test statistical hypotheses about scaled versions of the same system $g(s)$, then one might ask if the identification procedure used is equivariant with respect to the scalings. Problems of bayesian inference lead naturally to integral geometry on families such as F_g. A difficult question in general is that of the complete parametrization of a family. We will return to this later. Some of these questions are investigated in Krishnaprasad (1977) and Brockett and Krishnaprasad (1980).

Perhaps the best known example of a family of systems in the literature is a feedback family. It is obtained as follows. Consider a linear input-system with constant coefficients

$$\dot{x} = Ax + Bu$$

with m inputs and n states. The transformations

(a) $[A, B] \to [PAP^{-1}, PB]$; $P \in Gl(n)$;

(b) $[A, B] \to [A, PQ^{-1}]$, $Q \in Gl(m)$;

and

(c) $[A, B] \to [A - BK, B]$, $K \in L(m, n)$.

generate the feedback group. The collection of pairs $[\tilde{A}, \tilde{B}]$ obtained in this way from the given pair $[A, B]$ defines a family of input-systems. Given such an input-system, in trying to predict the effects of component failure, it is useful to 'embed' the problem in a family and examine what 'limiting' (or singular) systems are candidates for failed models (Martin 1979 a). The following coarse classification is useful.

(i) The system remains controllable but actuator failures lead to non-generic systems (for example, Kronecker invariants become 'singular').

(ii) The system suffers a loss of controllability and breaks up into an autonomous part and a controllable part.

(iii) The system parameters enter a range where there is a clear separation of time scales—a slow subsystem of low order, and a fast subsystem of low order (for example, high gain feedback effects (Young *et al.* 1977)).

Using basic algebro–geometric techniques, we show how one might 'predict' the structure of the failed system even in the absence of explicit information about the failure mode. In particular, this range of ideas is of relevance to certain families of systems obtained by linearization of non-linear systems. Meyer and Cicolani (1975) developed a formal structure for flight control systems with strong non-linearities, that includes a perturbation controller, of the form

$$\dot{x}(t, \lambda) = A(\lambda)x(t, \lambda) + B(\lambda)u(t, \lambda)$$

$$y(t, \lambda) = C(\lambda)x(t, \lambda)$$

with feedback law : $u(t, \lambda) = K(t, \lambda)x(t, \lambda)$.

The parameter λ of course depends on the flight regime or nominal trajectory. Our previous remarks on the feedback family apply to the design of a robust controller as λ varies over the flight envelope. Similar considerations also hold in problems of aircraft engine control (see, for example, DeHoff and Hall (1977)).

Other examples that lie within the framework of families of systems include the discrete families of the theory of jump processes (Brockett and Blankenship 1977) and systems with commensurate pure delays (Duncan 1979). Hazewinkel (1979 b) considers ideas on ring models which are yet another example of families of systems. A recurring problem in control applications is to approximate a complex model by simpler (finite-dimensional) models, which are constrained to lie in a family of models (specified for example by McMillan degree, Cauchy index, stability, etc.). Certain information-theoretic measures of approximation lead to variational problems on families of systems (Evans 1980, Evans and Krisnaprasad 1979).

In the next section we present a general framework for understanding families of systems. Some of these ideas have appeared in preliminary form in the conference papers (Martin 1979). Most of the differential geometric concepts and results we need may be found in Brockett and Krishnaprasad (1980) and Hermann and Krener (1977).

2. Families of systems : generalities

Consider linear systems with constant real coefficients of the form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. A, B and C are matrices of compatible dimensions. The collection $\Sigma_{n,m,p}$ of all such systems (or triples $[A, B, C]$) has the structure of the analytic manifold $\mathbb{R}^{n^2+n(m+p)}$.

Definition

A family of systems is a pair (Λ, ϕ) where Λ is a topological space and $\phi : \Lambda \to \Sigma_{n,m,p}$ is a continuous map. Further, if Λ is an analytic (resp. C^∞) manifold and ϕ is an analytic (resp. C^∞) map, then we have an analytic (resp. C^∞) family.

We shall often refer to either the map ϕ or the image im (ϕ) itself as a family. This should be clear from the context. Let $\Sigma_{n,m,p}{}^{r,0} \subset \Sigma_{n,m,p}$ denote the analytic manifold of completely reachable and completely observable systems (triples). We say that (Λ, ϕ) is a 'regular family' if im $(\phi) \subset \Sigma_{n,m,p}{}^{r,0}$. Otherwise we have a 'family with singularities'. Completely analogous definitions hold when we are concerned with systems over the complex field $\mathbb{C}$.

If the family defined by $\lambda \to \phi(\lambda) = [A(\lambda), B(\lambda), C(\lambda)]$ leaves $C(\lambda) =$ constant $= C_0$, then we can identify im (ϕ) with a subset of input-systems (or simply pairs $[A, B]$). Further, if the family is regular, then we can identify im (ϕ) with a subset of the analytic manifold $\Sigma_{n,m}{}^r$ of all controllable pairs. In other words, the map $\phi : \Lambda \to \Sigma_{n,m,p}{}^{r,0}$ induces a map $\tilde{\phi} : \Lambda \to \Sigma_{n,m}{}^{r}$ such that the following diagram commutes

$$\begin{array}{ccc} \Lambda & \xrightarrow{\quad \phi \quad} & \Sigma^{r,o}_{n,m,p} \\ & {\scriptstyle\tilde{\phi}}\searrow \qquad \swarrow{\scriptstyle\pi} & \\ & \Sigma^{r}_{n,m} & \end{array} \tag{2.2}$$

The projection π simply ‘ forgets ’ C.

For our purposes, the most important families of systems will be families that arise from the natural group actions of system theory (‘ group families ’). More general, but equally important, are the foliated families. Recall (Herman 1978) that an action of a Lie group G on a differentiable manifold M is a smooth map

$$\psi: \ G \times M \to M$$

$$(g, m) \to gm = \psi(g, m)$$

satisfying the following three points.

(a) For any $g \in G$, the maps $\psi_g: \ M \to M$ and $m \to gm$ are diffeomorphisms.

(b) For each $m \in M$, $g_1, g_2 \in G$, the relation $(g_1 g_2)m = g_1(g_2 m)$ holds.

(c) For each $m \in M$, the relation $em = m$ holds where e is the identity of G.

Associated with each point $m \in M$ is an ‘ orbit map ’

$$\psi^m: \ G \to M$$

$$g \to gm = \psi(g, m)$$

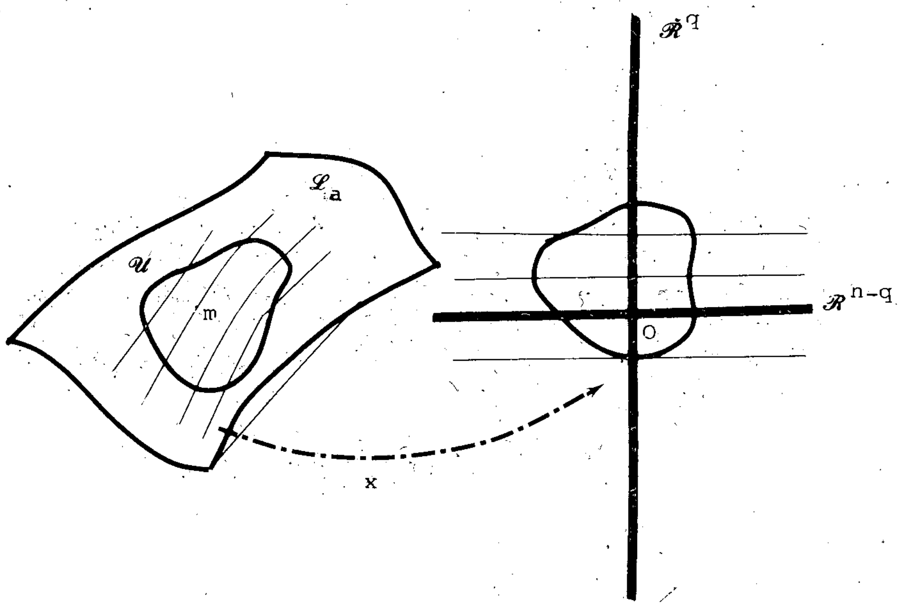
The image of ψ^m is known as the ‘ orbit ’ through m and is denoted $\mathcal{O}_m$. The structure of an orbit $\mathcal{O}_m$ is determined by the ‘ stabilizer ’ of m which is the closed subgroup $G_m \subset G$ defined by

$$G_m = \{g \in G : \ gm = m\}$$

Now each orbit $\mathcal{O}_m$ has the structure of a homogeneous space, G/G_m. Suppose $G_m = \{e\}$ for each $m \in M$. We then say that the action of G is ‘ free ’, in which case the manifold M is ‘ partitioned ’ into orbits $\mathcal{O}_m$ each of which looks like a copy of the group G, except that there is no canonical association of a point of

$\mathcal{O}_m$ with the identity e of G. Generalizing this concept of a decomposition of a manifold, we are lead to the notion of a ' foliation ' (Lawson 1974) :

> A smooth codimension-q foliation of an n-dimensional manifold M is a decomposition of M into a union of disjoint connected subsets $\{\mathscr{L}_\alpha : \alpha \in A\}$ called ' leaves ' of the foliation with the property that every point $m \in M$ has a nbhd U and a system of local coordinates $x = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ such that for each leaf $\mathscr{L}_\alpha$, the components of $U \cap \mathscr{L}_\alpha$ are described by the equations
>
> $$x^{n-q+1} = \text{constant}, \ldots, x^n = \text{constant}$$
>
> each leaf is then an ' immersed submanifold ' of M. The following diagram is helpful.



We now see how this circle of ideas leads to families of systems. Let G be a Lie group acting (on the left) smoothly on $\Sigma_{n,m,p}$

$$\psi : G \times \Sigma_{n,m,p} \to \Sigma_{n,m,p}$$

Then the orbit map, $\psi^{[A,B,C]}$ associated with a point $[A, B, C] \subset \Sigma_{n,m,p}$, defines a family of systems

$$\psi[A, B, C] : G \to \Sigma_{n,m,p}$$

$$g \to g[A, B, C] = \psi(g, A, B, C)$$

Here the group G itself serves as the parameter space Λ. From this point of view, a family is obtained by ' deformation ' of a given triple $[A, B, C]$ into its orbit $\mathcal{O}_{[A,B,C]}$. Similar considerations apply when G is restricted to act on $\Sigma_{n,m,p}{}^{r,0}$ or $\Sigma_{n,m}{}^{r}$. Families derived from foliations of $\Sigma_{n,m,p}$ are identified by defining the image of the parameter set to be a collection of leaves. We will

see later that together with a notion
necessary machinery to deal with
with families of systems.

We consider some examples.

Example 1

$\phi_1 : \Lambda = \mathbb{R}^4 \to \Sigma_{2,1,1}$

$\lambda = (p_0, p_1, q_0, q_1) \to ($

is the assignment of the standard co
ties (pole-zero cancellations) (Brock

$\phi_1{}^{-1}(\Sigma_{2,1,1}{}^{r,0}) \simeq \text{rat}\ (2) \equiv$

Example 2

$\phi_2 : \Lambda = \mathbb{R}^3 \times S^1 \to$ (comp

$(\lambda, \mu, \gamma, \theta) \to \frac{1}{s^2 + e^\lambda}$ | $e^\mu s$ … e^{μ}…

That this map ϕ_2 is onto may be se

Example 3

Fix $[A, B]$ a controllable pair

$\phi_3 = \psi^{[A,B]} : \mathscr{F} \to$

$(P, K, Q) \to$

Here $\mathscr{F}$ is the feedback group (se
$P \in Gl(n)$, $Q \in Gl(m)$, $K \in L(n; m)$ is a
family attached to the pair $[A, B]$.

Example 4

$\phi_4 : S^1$

λ

Here we view the family ϕ_4 as ta
maps (transfer functions) instead of

see later that together with a notion of ' foliation with singularities ' we have the necessary machinery to deal with most system-theoretic problems connected with families of systems.

We consider some examples.

Example 1

$$\phi_1: \Lambda = \mathbb{R}^4 \to \Sigma_{2,1,1}$$

$$\lambda = (p_0, p_1, q_0, q_1) \to \left(\begin{bmatrix} 0 & 1 \\ -p_0 & -p_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [q_0, q_1] \right)$$

is the assignment of the standard controllable form. This family has singularities (pole-zero cancellations) (Brockett and Krishnaprasad 1980)

$$\phi_1^{-1}(\Sigma_{2,1,1}^{r,0}) \simeq \text{rat}\ (2) \equiv \text{rational functions of degree 2}$$

Example 2

$$\phi_2: \Lambda = \mathbb{R}^3 \times S^1 \to (\text{compact, lossless, two-port networks})$$

$$(\lambda, \mu, \gamma, \theta) \to \frac{1}{s^2 + e^\lambda} \begin{vmatrix} e^\mu s & e \dfrac{(\mu+\gamma)}{2} [s \cos \theta + e^{\lambda/2} \sin \theta] \\ e \dfrac{\mu+\gamma}{2} [s \cos \theta - e^{\lambda/2} \sin \theta] & e^\lambda s \end{vmatrix}$$

That this map ϕ_2 is onto may be seen from Krishnaprasad (1979 a).

Example 3

Fix $[A, B]$ a controllable pair $\epsilon \Sigma_{n,m}^r$. Consider the map

$$\phi_3 = \psi^{[A,B]}: \mathscr{F} \to \Sigma_{n,m}^r$$

$$(P, K, Q) \to [P(A - BKP)P^{-1}, PBQ]$$

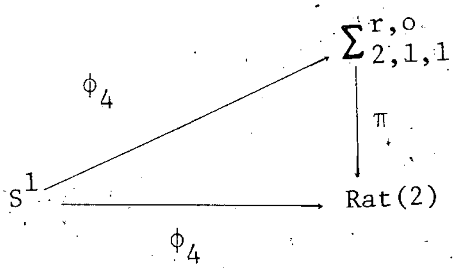
Here $\mathscr{F}$ is the feedback group (semidirect product of $Gl(n)$ and $Gl(m)$) and $P \in Gl(n)$, $Q \in Gl(m)$, $K \in L(n\ ;\ m)$ is an $m \times n$ matrix. $\psi^{[A,B]}$ defines the feedback family attached to the pair $[A, B]$.

Example 4

$$\phi_4: S^1 \to \text{rat}\ (2)$$

$$\lambda \to \frac{s \cos(\lambda) + \sin(\lambda)}{s^2 + 1}$$

Here we view the family ϕ_4 as taking values in a collection of input–output maps (transfer functions) instead of in a collection of triples. However, one can always find $\tilde{\phi}_4$ (a 'lift') to fill in the commutative diagram



(2.3)

π is the projection assigning transfer functions to triples. We might set $\tilde{\phi}^4$ equal to the standard controllable form. We will say more about this example later.

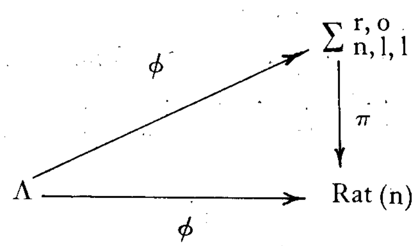
It is now possible to identify a list of basic problems associated with families of systems.

(A) Complete parametrization problem

Given a family (Λ, ϕ), determine the geometry of im (ϕ) completely. Further specify a set of coordinate charts for im (ϕ). This problem has its roots in fundamental investigations of the identification problem (Brockett 1976, Hazewinkel 1977, Byrnes and Hurt 1978, Brockett and Krishnaprasad 1980). In particular, the solution of the complete parametrization problem depends on knowing what the various topological and algebraic invariants of families are.

(B) The canonical-form problem

As we saw in Examples 2 and 4, one might specify families taking values in transfer functions instead of in triples. Typically, the canonical form question takes the following form : does there exist a (continuous) lifting (or canonical form) $\tilde{\phi}$ such that the following diagram commutes ?



(2.4)

The answer is yes in this case. However, if we replace rat (n) by some subset of multivariable transfer functions and $\Sigma_{n,1,1}{}^{r,0}$ by $\Sigma_{n,m,p}{}^{r,0}$, then this answer is in general no ! It depends on how 'twisted' the map π is over the image of ϕ. Even in the single-input–single-output case, if we replace $\Sigma_{n,1,1}{}^{r,0}$ by the restricted set of signature-symmetric triples satisfying $TA = A'T$ and

$Tb = c'$, where T is a signature m
(recall : every rational function a
1970)), then in general a lifting

$\tilde{\phi}$: Λ

does not exist.

In fact, in Example 4 the map

collection of signature-symmetric

functions in im (ϕ_4) has the struct
there is no lifting ! (See Byrnes (

(C) Closure problem

In general, im (ϕ) is not closed
The points in the boundary ∂ i
sequences of systems in the family
identification of systems with incr
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degenerate systems obtained as a
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im $(\phi_3) = \mathscr{O}_{[A,B]}$. In particular, [
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with any group-family we have t

(D) Investigate the stabilizer

This is very important, since
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Remark 1

The rest of this paper will be
programme represented by our
interesting and fairly general fami
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1979) and the significance of the p
is only now being appreciated.
problem in connection with a
functions). More generally, we h

(*a*) attach geometric (' invaria

(*b*) examine how these obje
(singularities, limits, etc.)

$Tb = c'$, where T is a signature matrix with 1s and -1s along the diagonal (recall : every rational function admits such a minimal realization (Brockett 1970)), then in general a lifting

$$\tilde{\phi} : \Lambda \rightarrow \Sigma_{n,1,1}{}^{r,0}\,(\mathrm{sym})$$

does not exist.

In fact, in Example 4 the map ϕ_4 is an embedding and one can verify that the collection of signature-symmetric triples $\left(\text{with } T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$ with transfer functions in im (ϕ_4) has the structure of the ' Mobius bundle ' over S^1 and hence there is no lifting ! (See Byrnes (1978) for this and many interesting remarks.)

(*C*) *Closure problem*

In general, im (ϕ) is not closed, and the problem is to determine its closure. The points in the boundary ∂ im (ϕ) have the interpretation as limits of sequences of systems in the family. Indeed, such sequences appear in recursive identification of systems with increasing sets of input–output data, as well as in adaptive control problems. The limiting systems may also be viewed as degenerate systems obtained as a consequence of failure of a given system in a family, in which case one would like to predict what characteristics (Kronecker invariants, McMillan degree, etc.) that the limiting systems might have. In this context the coarse classification mentioned in the Introduction is relevant.

In connection with the feedback family (see Example 3) we note that im $(\phi_3) = \mathcal{O}_{[A,B]}$. In particular, $[A, B]$ itself belongs to im (ϕ_3). In general, ϕ_3 is far from being an ' immersion ', and one would like to determine $\phi_3^{-1}([A, B])$. This preimage is simply the stabilizer $\mathscr{F}_{[A,B]}$. More generally, in connection with any group-family we have the problem.

(*D*) *Investigate the stabilizer*

This is very important, since the stabilizer determines the structure of the family (orbit).

Remark 1

The rest of this paper will be concerned with explicitly carrying out the programme represented by our list of problems with reference to certain interesting and fairly general families of systems. Since system-dependence on parameters can appear in surprisingly diverse ways, we expect such a programme to lead to some organizing principles. It should be pointed out that the complete parametrization problem is in general very difficult (see Segal 1979) and the significance of the problem to identification and adaptive control is only now being appreciated. Hazewinkel (1979) considered the closure problem in connection with approximating input–output maps (transfer functions). More generally, we have the following scheme :

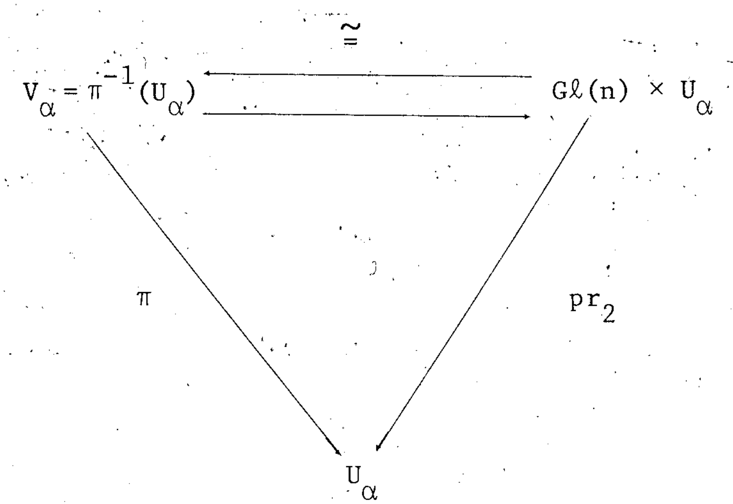
(*a*) attach geometric (' invariant ') objects to each member of a family ; and

(*b*) examine how these objects deform as we approach special points (singularities, limits, etc.) of a family.

The relevant objects will be clear from the context. Before we close this section, we point out some basic facts about the space $\Sigma_{n,m,p}^{r,0}$ of completely reachable and completely observable triples $[A, B, C]$. The group $Gl(n)$ acts on this manifold

$$\left.\begin{aligned} \psi : \ & Gl(n) \times \Sigma_{n,m,p}^{r0} \rightarrow \Sigma_{n,m,p}^{r,0} \\ & (P, A, B, C) \rightarrow [PAP^{-1}, PB, CP^{-1}] \end{aligned}\right\} \quad (2.5)$$

As a consequence of controllability (or observability) the action is 'free'. Since points of $\Sigma_{n,m,p}^{r,0}$ that are not on the same orbit have distinct transfer functions, the quotiant $M = \Sigma_{n,m,p}^{r,0}/Gl(n)$ is of interest. It has been shown (Hazewinkel and Kalman 1976, Byrnes and Hurt 1978) that $\Sigma_{n,m,p}^{r,0}$ can be covered by (locally trivial) neighbourhoods of the form $V_\alpha \simeq Gl(n) \times U_\alpha$ where U_α are neighbourhoods of the quotient. The Kronecker theory of canonical forms for systems played a significant role here. A previous result (1977) was that $(\Sigma_{n,m,p}^{r,0}, \pi, M)$ is a 'principal fibre bundle' (see Brockett and Krishnaprasad (1980), Hermann (1978) for an elaboration of this notion); this implies that the natural map $\pi : \Sigma_{n,m,p}^{r,0} \rightarrow M$, has fibres (preimages of points) that are diffeomorphic to $Gl(n)$. Further, the local triviality property (see Diagram (2.6)) holds.



(2.6)

What we would like to emphasize is that this result can be proved without resorting to canonical forms. This depends on the existence of a $Gl(n)$-invariant riemannian metric on $\Sigma_{n,m,p}^{r,0}$. Let N^r and N^0 be the $n \times nm$ and $np \times n$ matrices,

$$\left.\begin{aligned} N^r &= [B, AB, A^2B, \ldots, A^{n-1}B] \\ N^0 &= [C, CA, \ldots, CA^{n-1}] \end{aligned}\right\} \quad (2.7)$$

Then a riemannian metric can be defined on $\Sigma_{n,m,p}^{r,0}$ as a quadratic differential form

$$ds^2 = \text{tr}\,(N^0\, dA N^r N^{r'}\, dA' N^{0'}) + \text{tr}\,(dC N^r N^{r'}\, dC') + \text{tr}\,(dB' N^{0'} N^0\, dB) \quad (2.8)$$

We omit verification that ds^2 is
bility and observability), and is
that each orbit $\mathcal{O}_{[A,B,C]}$ is closed
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3. Families of input systems

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3.1. *Commuting vector fields in* $\mathbb{R}$

Let M^n be a C^∞ (analytic) ma
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$[X, Y] = \sum$

Given any subset $\{X_\alpha\} \subset \mathcal{U}(M$
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other integral submanifold of $\mathcal{L}$.
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Nagano 1966).

We omit verification that ds^2 is non-degenerate (a consequence of controllability and observability), and is $Gl(n)$-invariant. A simple calculation shows that each orbit $\mathscr{O}_{[A,B,C]}$ is closed in $\Sigma_{n,m,p}{}^{r,0}$. Now using geodesic neighbourhoods with respect to eqn. (2.8), one can establish the Hausdorff property of the quotient topology. In order to finish the proof, we need to conformally modify the metric into a 'complete metric'. (For this procedure, see the theorem due to Nomizu–Ozeki in Hermann (1978, p. 288). Details of the proof will appear elsewhere.) The point of this digression is to indicate that the space $\Sigma_{n,m,p}{}^{r,0}$ carries interesting structures such as $Gl(n)$-invariant riemannian structures. For historical remarks concerning this moduli problem, see Delchamps (1980).

In the next two sections we take up the geometric characterization of specific families.

3. Families of input systems

Here we address the specific question : what does the family of controllable systems of the form

$$\dot{x} = Ax + Bu, \quad u \in \mathbb{R}^m, \quad x \in \mathbb{R}^n \tag{3.1}$$

where A is fixed, look like ? A related problem is to understand how uncontrollable systems appear via degenerating a family of controllable systems. We show how both these questions (of parametrization and closure) can be answered via the study of certain foliations with singularities. The single input case yields very explicit results. In this case, the geometry of cyclic vectors associated with A has to be understood and commuting vector fields in $\mathbb{R}^n$ play an important part.

3.1. *Commuting vector fields in* $\mathbb{R}^n$

Let M^n be a C^∞ (analytic) manifold of dimension n. Let $\mathscr{U}(M^n)$ denote the set of C^∞ (analytic) vector fields together with the Lie algebra structure. Recall (Hermann 1978) that for a local coordinate system, $x = (x_1, \ldots, x_n)$ and associated basis vector fields, $\partial/\partial x_i$, $i = 1, 2, \ldots, n$, the Lie bracket of two vector fields $X = \sum_i f_i\,(\partial/\partial x_i)$ and $Y = \sum_i g_i\,(\partial/\partial x_i)$ is given by

$$[X, Y] = \sum_{i,j} \left(f_i \frac{\partial}{\partial x_i} g_j - g_i \frac{\partial}{\partial x_i} f_j \right) \frac{\partial}{\partial x_i} \tag{3.2}$$

Given any subset $\{X_\alpha\} \subset \mathscr{U}(M^n)$, we denote the Lie algebra generated by this subset as $\mathscr{L}$. It simply consists of finite linear combinations of elements of the form $[X^{\alpha_1}[X^{\alpha_2} \ldots [X^{\alpha_{k-1}}, X^{\alpha_k}] \ldots]]$. At any point, $x \in M^n$, the elements of $\mathscr{L}$ span a subspace of the tangent space $TM_x{}^n$. We denote this subspace as $F\{X^\alpha\}_x$. Now a connected submanifold $N \subset M^n$ is said to be an 'integral submanifold' of M^n if at each $x \in N$, the tangent space to N at $x \in TN_x \subset F\{X^\alpha\}_x$. N is a 'maximal integral submanifold' of M^n if it is not properly contained in any other integral submanifold of $\mathscr{L}$. The existence of maximal integral submanifolds is guaranteed in two cases (Hermann and Krener 1977, Hermann 1962, Nagano 1966).

Theorem (*Frobenius*)

If the dimension of $F\{X^\alpha\}_x = k$ for every $x \in M^n$, then there exists a partition of M into maximal integral submanifolds of $\mathscr{L}$, all of dimension k (a foliation of codimension $n-k$).

Theorem (*Hermann–Nagano*)

If the distribution $\mathscr{L}$ is analytic, then there exists a partition of M into maximal integral submanifolds of $\mathscr{L}$ of varying dimensions. The dimension of $F\{X^\alpha\}_x$ is itself constant on each submanifold of the partition and is equal to the dimension of that submanifold. The partition defines a foliation with singularities.

In this paper, we are solely concerned with the analytic case. Let $M^n = \mathbb{R}^n$. Consider a fixed ‘ cyclic matrix ’ A. We will show that a pair $[A, x]$ may be viewed as a controllable pair or as a system obtained by degenerating a family of controllable systems.

Let $\{X^0, X^1, \ldots, X^{n-1}\}$ be the set of analytic vector fields in $\mathbb{R}^n$ defined by

$$X^k = \sum_i \sum_j A_{ij}{}^k x_j \frac{\partial}{\partial x_i} \tag{3.3}$$

Here $A_{ij}{}^k$ is the (i, j)th element of the matrix power A^k. The vector fields X^k and X^j commute (i.e. $[X^k, X^j] = 0$) for all k, j $\{0, 1, 2, \ldots, n-1\}$. Hence the Lie algebra $\mathscr{L}$ generated by $\{X^0, \ldots, X^{n-1}\}$ is finite dimensional. Suppose we denote as F_x the subspace of the tangent space of $TM_x{}^n \simeq \mathbb{R}^n$ spanned by the elements of $\mathscr{L}$ at $x \in \mathbb{R}^n$. Then $F_x = \text{span}\,\{x, Ax, \ldots, A^{n-1}x\}$. Since A is cyclic, $\dim F_x = n$ on a dense open subset of $\mathbb{R}^n$ denoted as C_A. Thus $C_A = \bigcup_\alpha N_\alpha{}^n$ where each $N_\alpha{}^n$ is a maximal integral manifold of dimension n and is thus a connected component of C_A. There are several steps involved in determining the geometry of the manifold C_A of cyclic vectors of A.

First, note that each X^k is a complete vector field, since integral curves of X^k are of the form

$$\{\exp(A^k t)x : \; t \in \mathbb{R}\}$$

Let $x \in C_A$ and $N(x)$ denote the maximal integral submanifold of $\mathscr{L}$ passing through x equal the connected components of C_A containing x. Then the exponential Lie group generated by the vector fields $X^0, X^1, \ldots, X^{n-1}$ defines an abelian action

$$\psi : \mathbb{R}^n \times N(x) \to N(x)$$

$$((t_0, t_1, \ldots, t_{n-1}), z) \to \exp\left(\sum_{k=0}^{n-1} t_k A^k\right) z \triangleq \psi(t, z)$$

Now for $z \in N(x)$, define the orbit map

$$\psi^z : \mathbb{R}^n \to N(x)$$

$$t \to \psi(t, z)$$

The jacobian

$$d\psi^z(t) : \; \mathbb{R}^n \to \mathbb{R}^n$$

$$h \to (y \;\; Ay \;\; \ldots \;\; A^{n-1}y)h$$

where $y = \exp\left(\sum_{k=0}^{n-1} t_k A^k\right) z \in C_A$,
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The stabilizer of the abelian actio

$H_z = \{(t_0, t_1, \ldots$

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Theorem 1

Each connected component N
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Remark 2

The integer k depends only o
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3.2. *The connectivity of* C_A

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where $y=\exp\left(\sum_{k=0}^{n-1} t_k A^k\right) z \in C_A$, is clearly onto. Thus ψ^z is a local diffeomorphism. Repeated application of the implicit function theorem yields

$$\text{im}\,(\psi^z)=N(x)$$

The stabilizer of the abelian action is the subgroup of $\mathbb{R}^n$ defined by

$$H_z=\{(t_0, t_1, \ldots, t_{n-1})=t\in\mathbb{R}^n:\ \psi(t, z)=z\}$$

It is an immediate consequence of the 'local freeness' of the action ($d\psi_z$ onto), that the stabilizer subgroup is 'discrete'. It is a well-known result in the theory of topological groups that any discrete subgroup of $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^k$ and equal to the product of several copies of the integers (Arnold 1978). In the present case, the integer k is locally constant and hence constant on the connected component $N(z)$, and we have proved the following theorem.

Theorem 1

Each connected component $N(z)$ of C_A is diffeomorphic $T^k\times\mathbb{R}^{n-k}$, where T^k denotes the k-torus.

Remark 2

The integer k depends only on A and is equal to the number of complex conjugate pairs of eigenvalues of A. It is thus constant on C_A. We omit the verification of this remark.

Remark 3

The proof of the theorem is along the lines of the invariant-tori theorem of classical mechanics (Arnold 1963, 1978). In fact, we were led to the theorem by way of earlier investigation in symplectic mechanics (Krishnaprasad 1979). We will return to this towards the end of this section.

Remark 4

We see that $\mathbb{R}^n$ is partitioned into leaves of a foliation with singularities. The manifold C_A is the union of a maximal dimension leaves (integral submanifolds). In the next subsection, we count the number of such leaves. The leaves of lower dimension (singularities) correspond to uncontrollable systems.

3.2. *The connectivity of C_A*

In this subsection, we explore further the group-theoretic aspects of our foliation. Recall that $C_A\triangleq\{x^n|x, Ax, \ldots, A^{n-1}x \text{ span } \mathbb{R}^n\}$ is open dense in $\mathbb{R}^n$. In general if y C_A, then there exists $P_y\in Gl(n)$ such that

$$\left.\begin{aligned} P_yAP_y^{-1}&=\tilde{A}\\ P_yy&=e_n\end{aligned}\right\}\qquad(3.4)$$

where $\tilde{A}$ is the companion form (unique rational canonical form) associated with A and $e_n=(0, 0, \ldots, 1)'$ a standard basis vector in $\mathbb{R}^n$. Now, if $z\in C_A$, then

and

$$\left.\begin{aligned} P_z^{-1}P_yA(P_z^{-1}P_y)^{-1}&=A\\ P_z^{-1}P_y\cdot y&=z\end{aligned}\right\}\qquad(3.5)$$

Thus $T = P^{-1}P_y \in H_A$, the subgroup of $Gl(n)$ which 'stabilizes' A under similarity. We see that H_A acts on C_A

$$\left.\begin{aligned} \gamma:\ & H_A \times C_A \to C_A \\ & (T, y) \to Ty = z \end{aligned}\right\} \tag{3.6}$$

It can be verified that H_A acts freely (a consequence of controllability). Thus C_A carries a group structure—exactly the same as that associated with the representation of C_A as a union of maximal dimension leaves (Theorem 1). Recall that since A is cyclic, H_A is the abelian group of units in the ring of polynomials in A generated by $I, A, A^2, \ldots, A^{n-1}$ (see Gantmakher 1959, p. 223). Formally

$$H_A = P = \sum_{i=0}^{n-1} \alpha_i A^i \,|\, \alpha_i \in \mathbb{R} \text{ and } P \text{ invertible} \tag{3.7}$$

Since C_A is diffeomorphic to H_A, we have the equality

$$\#(\text{connected components of } C_A) = \#(\text{connected components of } H_A) = \mu_A$$

If we denote $P_\alpha = \sum_{i=0}^{n-1} \alpha_i A^i$ and $P_\beta = \sum_{i=0}^{n-1} \beta_i A^i$ two distinct points of H_A, then a continuous curve in $\mathbb{R}^n$ joining $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ and $\beta = (\beta_0, \beta_1, \ldots, \beta_{n-1})$ fails to be a deformation of P_α into P_β if and only if it contains an intermediate point $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ such that P_γ is singular. Now by the spectral mapping theorem, the spectrum of P_γ is the set

$$\{\gamma(\lambda_1), \ldots, \gamma(\lambda_n)\}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of A and $\gamma(s) = \gamma_0 + \gamma_1 s + \ldots + \gamma_{n-1}s^{n-1}$. Thus P_γ is non-singular if and only if $\gamma(\lambda_i) \neq 0$ for $\lambda_i \in$spectrum (A). But this is the same as saying that $\gamma(s)$ and $p(s)$ are characteristic polynomials of A but do not have any common factors. We have thus established a one-to-one correspondence between H_A and the subspace of rat (n), defined by

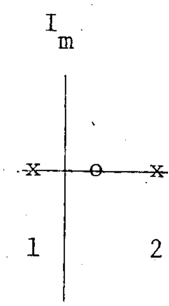
$$M_p = \frac{q_0 + q_1 s + \ldots + q_{n-1}s^{n-1}}{p(s)} : \ q(s) \text{ and } p(s) \text{ relatively prime}$$

Here $p(s)$ is fixed. To determine the connectivity of M_p (which is equal to the connectivity of H_A), we now use an idea first introduced in Brockett (1976), namely deformation of pole-zero patterns. A pole-zero pattern determines a rational function $q(s)/p(s)$ up to scale-factor. The difference here is that the poles are fixed. The following remarks apply

(a) Complex poles and zeros do not create obstructions to deformations.

(b) A rational function with complex zeros or real zeros of even multiplicity can be continuously deformed into one where these zeros appear at ∞.

(c) One can verify that if $p(s)$
$q(s)/p(s)$ can be deformed
form



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(d) Now if $r \neq 0$, then in order t
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In summary, we have the foll

Theorem 2

The space M_p of rational fu
polynomial $p(s)$ has 2^r connected
distinct poles.

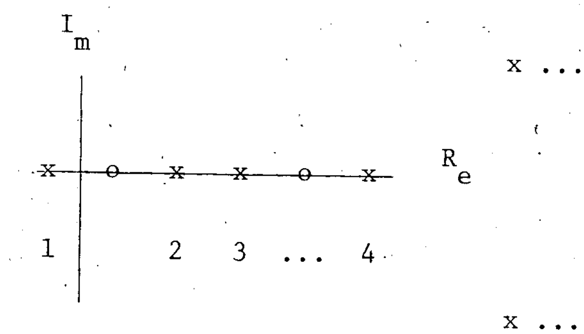
Combining this result with the
and the correspondence between

Theorem 3

The family of controllable pai
disjoint union

where r = number of real distinct
morphic to the product $T^k x^{n-k}$, w
pairs of A.

(*c*) One can verify that if $p(s)$ has r real zeros (ignoring multiplicities) then $q(s)/p(s)$ can be deformed continuously into a pole-zero pattern of the form



where at most one zero appears in the cell defined by a pair of adjacent poles. There are exactly 2^{r-1} such standard patterns. The zeros in any given pattern are trapped zeros, i.e. a zero cannot be moved from one cell to another without passing through a pole-zero cancellation. Thus each of the standard patterns represents a distinct homotopy class of pole-zero patterns.

(*d*) Now if $r \neq 0$, then in order to change the sign of the scale factor one has to allow the coefficient of the highest power of s in $q(s)$ to vanish at some point. This requires that a trapped zero be moved to infinity and hence cannot be accomplished without pole-zero cancellation. Hence, when $r \geqslant 1$ rational functions with scale factors of different signs cannot be deformed into each other. When $r = 0$, there is no obstruction to the deformation of zeros arising from the relative primeness conditions. Hence in this case M_p is connected.

In summary, we have the following theorem.

Theorem 2

The space M_p of rational functions of degree n and fixed denominator polynomial $p(s)$ has 2^r connected components where r is the number of real distinct poles.

Combining this result with the previous remarks about the geometry of H_A and the correspondence between H_A and M_p we have another theorem.

Theorem 3

The family of controllable pairs $[A, b]$ for fixed A is diffeomorphic to the disjoint union

$$\bigcup_{j=1}^{2^r} L_j$$

where r = number of real distinct eigenvalues of A and each leaf L_j is diffeomorphic to the product $T^k x^{n-k}$, where k is the number of complex eigenvalues pairs of A.

3.3. *Closures and limiting systems*

It is clear that the lower dimensional (singular) leaves of the foliation with singularities defined by eqn. (3.3), correspond to uncontrollable systems. To obtain these as limits of controllable systems, one notices that any $y \in \mathbb{R}^n$ has a representation

$$y = \sum_{i=0}^{n-1} \alpha_i A^i x \tag{3.8}$$

where x is a cyclic vector of A. Hence

$$y = \left(\sum_{i=0}^{n-1} \alpha_i A^i \right) x$$

Here, $P = \sum_{i=0}^{n-1} \alpha_i A^i \in h_A$, the set of matrices that commute with A. Further, y is a cyclic vector if and only if P is non-singular (which holds if and only if $\alpha(s)$ and $p(s)$ do not have common factors). Thus given a pair $[A, x]$, we can generate a sequence of controllable pairs degenerating to an uncontrollable pair $[A, y]$ in the following way.

(i) Find the (unique) α_i in eqn. (3.8) and define $\alpha(s)/p(s)$ of degree $< n$.

(ii) Construct a sequence of rational functions of degree n of the form

$$\frac{\alpha^{(k)}(s)}{p(s)}$$

where $\alpha^{(0)}(s) = 1$ and such that

$$\alpha^{(k)}(s) \rightarrow \alpha(s) \text{ as } k \rightarrow \infty$$

It can easily be verified that this can always be done. This implies that the sequence of controllable systems, $[A, y^{(k)}]$ defined by

$$y^{(0)} = x$$

and

$$y^{(k)} = \left(\sum_{i=0}^{n-1} \alpha_i^{(k)} A^i \right) x$$

has the property that $[A, y^{(k)}] \rightarrow [A, y]$.

Khadr and Martin (1980) established closure results for $Gl(n)$ families which yield uncontrollable systems in the limit. Our own calculations can be refined much further to determine the invariants of uncontrollable systems.

Remark 5

Theorem 3 has several consequences for families of systems. Consider the group action (by the additive abelian group $\mathbb{R}^n$)

$$\psi : \mathbb{R}^n \times \Sigma_{n,1}{}^r \rightarrow \Sigma_{n,1}{}^r$$

$$((t_0, t_1, \ldots, t_{n-1}), [A, b]) \rightarrow \left[A, \exp\left(\sum_{i=0}^{n-1} t_i A^i \right) b \right]$$

From our previous remarks, the action is locally free (discrete stabilizer everywhere on $\Sigma_{n,1}{}^r$), and hence defines a foliation of $\Sigma_{n,1}{}^r$. Theorem 3

describes the structure of the leav
the leaves are parameterized b
complete parameterization probl
family defined by this $\mathbb{R}^n$-action.

Remark 6

We were led to these ideas
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Theorem 4

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Further, on rat $(n, 0)$, $m = 0$ and t

Remark 7

The statement about the co
(1976). The foliation was presen
So far in this section, we ha
This is possible as the stabilizer
similarity is abelian and is of dime
groups and the families have mor

4. Feedback families

We have already seen how a
wealth of information about fa
families. Consider the group $\mathscr{F}$

$$\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}; \quad P \in Gl$$

We call $\mathscr{F}$ the feedback group
dimension $n^2 + nm + m^2$. From E

$$\psi : \mathscr{F} \times \Sigma_{n,m}{}^r -$$

$$\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}, [A, B] -$$

describes the structure of the leaves. One consequence of this structure is that the leaves are parameterized by 'theta-functions' (Hancock 1958). The complete parameterization problem is in this sense solved for each (regular) family defined by this $\mathbb{R}^n$-action.

Remark 6

We were led to these ideas by investigations in analytical mechanics (Krishnaprasad 1979). The invariant-tori theorem says that if a hamiltonian system has a full set of n integrals of motion satisfying certain regularity conditions, then the phase space (of dim $=2n$) is partitioned into level manifolds by setting the integrals to various constant values. The level manifolds are products of tori and lines. The properties of certain dynamical systems were investigated previously (Brockett and Krishnaprasad 1980). Those were indeed the generators of families such as F_g in the Introduction. More completely we may state the next theorem.

Theorem 4

The analytic manifold rat (n) of proper rational functions of degree n has $(n+1)$ connected components distinguished by the Cauchy index. Each connected component rat (p, q) admits a foliation whose leaves are diffeomorphic to $T^m \times \mathbb{R}^{n-m}$ where m is constant on an open set. Each leaf is a connected component of the 'level manifold' obtained by setting $p_i=$ constant $=c_i$, $i=0, 1, 2, \ldots, n-1$ in $p(s)=s^n+p_{n-1}s^{n-1}+ \ldots +p_1 s+p_0$. Further, on rat $(n, 0)$, $m=0$ and the foliation is a fibration.

Remark 7

The statement about the connectivity of rat (n) was given in Brockett (1976). The foliation was presented in Brockett and Krishnaprasad (1980).

So far in this section, we have used abelian actions to great advantage. This is possible as the stabilizer H_A of a matrix for the action of $Gl(n)$ via similarity is abelian and is of dimension n. More generally, one is led to solvable groups and the families have more intricate structure.

4. Feedback families

We have already seen how a deeper study of the stabilizer H_A leads to a wealth of information about families. This is even more true of feedback families. Consider the group $\mathscr{F}$ of non-singular matrices of the form

$$\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}; \quad P\in Gl(n), \quad Q\in Gl(m), \quad K\in L(n\,;\,m)$$

We call $\mathscr{F}$ the feedback group and it is a closed subgroup of $Gl(n+m)$ of dimension n^2+nm+m^2. From Example 3 of § 2, we have the action

$$\psi : \mathscr{F} \times \Sigma_{n,m}{}^r \to \Sigma_{n,m}{}^r$$

$$\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}, [A, B] \to [P^{-1}AP+P^{-1}BK, P^{-1}BQ] \tag{4.1}$$

We call every orbit of the action, a feedback family and define this by the orbit map

$$\psi^{[A,B]}: \mathscr{F} \to \Sigma_{n,m}{}^{r}$$

$$\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix} \to [P^{-1}AP + P^{-1}BK,\ P^{-1}BQ]$$

Associated with each $[A, B]$ pair one has the following ordered set of subspaces

$$\mathscr{B}_0 \subseteq \mathscr{B}_1 \subset \dots \subset \mathscr{B}_{n-1} \tag{4.2}$$

where we have the recursive definition

$$\left.\begin{aligned} \mathscr{B}_0 &= \operatorname{im}(B) = \text{column space of } B \\ \mathscr{B}_{i+1} &= \mathscr{B}_i + A\mathscr{B}_i\,; \quad i = 0, 1, 2, \dots, n-2 \end{aligned}\right\} \tag{4.3}$$

Since $\mathscr{B}_{n-1}$ is simply the column space of the matrix

$$N^r \triangleq [B, AB, \dots, A^{n-1}B] \tag{4.4}$$

it follows from the standard criterion of controllability that whenever $[A, B]$ is a controllable pair, $\mathscr{B}_{n-1} = \mathbb{R}^n$. We then say that (4.2) defines a 'filtration' of $\mathbb{R}^n$ and the ordered n-tuple, $(\mathscr{B}_0, \mathscr{B}_1, \dots, \mathscr{B}_{n-1})$ is called the 'flag' associated with the pair $[A, B]$. We denote this as $\mathscr{V}_{[A,B]}$. Brunovsky (1970) noticed that transformations of the types, (i) $[A, B] \to [A + BK, B]$ and (ii) $[A, B] \to [A, BQ]$ leave the flag $\mathscr{V}_{[A,B]}$ invariant. Further, under $Gl(n)$ action, $[A, B] \to [PAP^{-1}, PB]$

$$(\mathscr{B}_0, \mathscr{B}_1, \dots, \mathscr{B}_{n-1}) \to (P\mathscr{B}_0, P\mathscr{B}_1, \dots, P\mathscr{B}_{n-1})$$

or more compactly

$$\mathscr{V}[PAP^{-1}, PB] = P_{\mathscr{V}[A,B]} \tag{4.5}$$

Equation (4.5) is very basic since it shows the assignment of a flag to a pair $[A, B]$ is well behaved with respect to the feedback group action. Let $l_i = \dim(\mathscr{B}_i)$, $i = 0, 1, \dots, n-1$. Associate with the sequence, $l_0 \leqslant l_1 \leqslant \dots \leqslant l_{n-1} = n$, the integers, $\sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant \sigma_n$ defined by

$$\begin{aligned} \sigma_1 &= l_0 \\ \sigma_i &= l_{i-1} - l'_{i-2}, \quad i = 2, \dots, n \end{aligned} \tag{4.6}$$

Clearly, $\sigma_1 + \sigma_2 + \dots + \sigma_n = n$ and the integers, $(\sigma_1, \dots, \sigma_n)$ determine a partition of the integer n. Clearly l_i and hence σ_i are invariant under the feedback group. That the partition $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a 'complete invariant' is the content of Brunovsky's (1970) main theorem in the language of canonical forms.

Theorem (*Brunovsky*)

Given $[A, B]$, a controllable pair, associate a partition σ of n as above. To this partition associate its dual (in the sense of Young diagrams) partition, $\rho = (\rho_1, \dots, \rho_{\sigma_1})$ where the Kronecker invariants ρ_i are ordered $\rho_1 \geqslant \rho_2 \geqslant \dots \geqslant \rho_{\sigma_1}$.

Then, on each orbit $\mathscr{O}_{[A,B]}$ of the fe
form

$A_\rho = \begin{bmatrix} J_{\rho_1} & 0 \\ 0 & J_\rho \\ \vdots \\ 0 \dots \end{bmatrix}$

$J_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots \\ 0 \dots \\ 0 \dots \end{bmatrix}$

and each E_k is a $\rho_k \times m$ matrix wit
equal to unity.

Corollary

#feedback orbits

The association of dual partition
standard (see, for example, Kalma
algebraic flavour of Brunovsky
geometry of the orbit $\mathscr{O}_{[A,B]}$ in
represent each orbit as a homogen
group. Specifically

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where $\mathscr{F}_{[A,B]}$ is the stabilizer of
belong to the same orbit, then the s
of $\mathscr{F}$) to $\mathscr{F}_{[\bar{A},\bar{B}]}$. In particular, it
associated with the Brunovsky car
that $\mathscr{F}_{[A_\rho, B_\rho]}$ consists of matrices

$P = \begin{bmatrix} P_{11} & P_{12} \dots \\ P_{21} & P_{22} \dots \\ \vdots \\ P_{\sigma_1 1} \dots \end{bmatrix}$

$P_{ij} = \begin{bmatrix} \alpha_{ij} \\ & \alpha_{ij} \\ && \ddots \\ &&& \alpha_{ij} \end{bmatrix}$

$P_{ij} = 0$ if $\rho_i > \rho_j$

Then, on each orbit $\mathcal{O}_{[A,B]}$ of the feedback group there is a canonical pair of the form

$$A_\rho = \begin{bmatrix} J_{\rho_1} & 0 \cdots\cdots & 0 \\ 0 & J_{\rho_2} \cdots\cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots\cdots 0 & J_{\rho_{\sigma_1}} \end{bmatrix}, \quad B = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_{\sigma_1} \end{bmatrix}$$

$$J_k = \begin{bmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \ddots \\ 0 & \cdots\cdots\cdots & 1 \\ 0 & \cdots\cdots\cdots & 0 \end{bmatrix} \text{ is a } k \times k \text{ matrix}$$

and each E_k is a $\rho_k \times m$ matrix with all elements zero except $(E_k)_{\rho_k, k}$ which is equal to unity.

Corollary

$$\#\text{feedback orbits} = \#\text{partitions of } n = \text{finite}$$

The association of dual partitions to a given partition via Young diagrams is standard (see, for example, Kalman (1971)). It is interesting to compare the algebraic flavour of Brunovsky (1970) and the first investigation of the geometry of the orbit $\mathcal{O}_{[A,B]}$ in Brockett (1977). Brockett's idea was to represent each orbit as a homogeneous space (Hermann 1970) of the feedback group. Specifically

$$\mathcal{O}_{[A,B]} = \mathscr{F}/\mathscr{F}_{[A,B]} \tag{4.7}$$

where $\mathscr{F}_{[A,B]}$ is the stabilizer of $[A, B]$. Note that, if $[A, B]$ and $[\tilde{A}, \tilde{B}]$ belong to the same orbit, then the stabilizer $\mathscr{F}_{[A,B]}$ is conjugate (as a subgroup of $\mathscr{F}$) to $\mathscr{F}_{[\tilde{A},\tilde{B}]}$. In particular, it is easier to compute the stabilizer $\mathscr{F}_{[A_\rho, B_\rho]}$ associated with the Brunovsky canonical form of an orbit. Brockett showed that $\mathscr{F}_{[A_\rho, B_\rho]}$ consists of matrices $\begin{bmatrix} P & 0 \\ K & Q \end{bmatrix}$ of the form

$$P = \begin{bmatrix} P_{11} & P_{12} \ldots P_{1\sigma_1} \\ P_{21} & P_{22} \ldots P_{2\sigma_1} \\ \vdots & \ddots \\ P_{\sigma_1 1} & \cdots\cdots P_{\sigma_1\sigma_1} \end{bmatrix}, \quad P_{ij} \text{ a } \rho_i \times \rho_j \text{ matrix}$$

$$P_{ij} = \begin{bmatrix} \alpha_{ij} & & \\ & \alpha_{ij} & \\ & & \ddots \\ & & & \alpha_{ij} \end{bmatrix} \quad \text{if } \rho_i = \rho_j$$

$$P_{ij} = 0 \quad \text{if } \rho_i > \rho_j$$

and

$$P_{ij} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} & \cdots & \delta_{ij} & \cdots & 0 \\ \vdots & & \ddots & & \ddots & \\ 0 & \alpha_{ij} & & \ddots & & 0 \\ \vdots & & \ddots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{ij} & \beta_{ij} & \delta_{ij} \end{bmatrix}, \quad \rho_1 < \rho_j$$

$$K = B_\rho{}^{\#}(PA_\rho - A_\rho P) + \tilde{K}$$

where $\#$ denotes the generalized inverse and $\tilde{K} = \left[\dfrac{0}{K^*}\right]$ with K^* being $(m-\sigma_1) \times n$ and arbitrary, and finally

$$Q = \left[\begin{array}{cccc|c} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\sigma_1} & \\ & \ddots & & & \\ \alpha_{12} & & \ddots & & \\ \vdots & & & & \\ \alpha_{\sigma_1 1} & \cdots & \cdots & \alpha_{\sigma_1\sigma_1} & \\ \hline & & * & & ** \end{array}\right]$$

the elements marked with an asterisk are arbitrary and those having a double asterisk are arbitrary and invertible. It is now a straightforward calculation to show that

$$\dim \mathscr{F}_{[A\rho, B\rho]} = (n+m)(m-\sigma_i) + \sum_{\rho_1 \geq \rho_j} (\rho_i + 1 - \rho_j) \tag{4.8}$$

Example 5

In the single-input case, $m = \sigma_1 = 1$, and $\rho_1 = n$. Thus $\dim \mathscr{F}_{[A,B]} = 1$ and the stabilizing matrices are of the form αI_n; $\alpha \neq 0$.

It is now standard (Hermann 1978) that the dimension of the orbit $\mathscr{F}_{[A,B]}$ as a homogeneous space is given by

$$\begin{aligned} \dim \mathscr{O}_{[A,B]} &= \dim \mathscr{F} - \dim \mathscr{F}_{[A,B]} \\ &= n^2 + nm + m^2 - \dim \mathscr{F}_{[A,B]} \end{aligned} \tag{4.9}$$

and one can use formula (4.8).

Now, notice that from the point of view of deformation of systems it is important to know whether a pair $[A, B]$ may be continuously deformed into a pair $[\tilde{A}, \tilde{B}]$ using the operations of the feedback group. Brunovsky's (1970) theorem is not sufficient to answer this question because the feedback group is not a connected Lie group. In fact $\mathscr{F}$ can be seen to have four components (Brockett 1977). In this sense, it is more natural to work with the connected component of the identity in $\mathscr{F}$ determined by the determinantal restriction $\det(P) > 0$, $\det(Q) > 0$. We call this group $\mathscr{F}^+$. One now determines the number of connected components of the stabilizer $\mathscr{F}_{[A,B]}$ using the explicit

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representation above. The num
number of components of $\mathscr{F}$ divi
the stabilizer intersects non-trivi

Theorem (*Brockett* 1977)

$\mathscr{O}_{[A,B]}$ is connected unless the
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of the flag partition.

Given any partition $\sigma = (\sigma_1, \ldots$
can define $Fl(\sigma_1, \ldots, \sigma_n)$ to be the

$$L_1 \subseteq L_2 \subseteq \ldots$$

and

$$\dim (L$$

Given two flags $(L_1, \ldots, L_n)$ and
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$$\tilde{L}_i = T$$

(choose a basis for L_1, extend this
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$$\begin{bmatrix} A_1 \\ \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where each A_{ij} is a $\sigma_i \times \sigma_j$ matrix representation above. The number of components of the orbit equals $\#$, the number of components of $\mathscr{F}$ divided by the number of components of $\mathscr{F}$ that the stabilizer intersects non-trivially. We now have the following theorem.

Theorem (*Brockett* 1977)

$\mathcal{O}_{[A,B]}$ is connected unless the n_i are all of the same parity (i.e. all even or all odd) in which case it consists of two connected components.

Brockett (1977) used these calculations to refine Brunovsky's (1970) normal form to determine when two pairs $[A, B]$ and $[\tilde{A}, \tilde{B}]$ are on the same orbit of $\mathscr{F}^+$. His result uses Hermann's (1978) lemma in an interesting way.

In our discussions so far we have seen two partitions: the Kronecker partition $(\rho_1, \rho_2, \ldots, \rho_{\sigma_1})$ of n, and the flag partition $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of n, which are dual to each other. We have seen the role played by the Kronecker partition in determining the dimension, connectivity, etc., of the $\mathcal{O}_{[A,B]}$. If the elements of the Kronecker partition are as equal as possible, the corresponding orbit has the largest dimension among all feedback families in $\Sigma_{n,m}{}^r$ and as such is considered to be the generic family. Now we proceed to investigate the role of the flag partition.

Given any partition $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, of the integer n, we can define $Fl(\sigma_1, \ldots, \sigma_n)$ to be the set of all n-tuples $(L_1, L_2, \ldots, L_n)$ where

$$L_1 \subseteq L_2 \subseteq \ldots \subseteq L_n = \mathbb{R}^n, \quad \dim (L_1) = \sigma_1$$

and

$$\dim (L_i) = \sigma_i + \dim (L_{i-1})$$

Given two flags $(L_1, \ldots, L_n)$ and $(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_n)$ belonging to $Fl(\sigma_1, \ldots, \sigma_n)$, there is an element $T \in Gl(n)$ such that

$$\tilde{L}_i = TL_i, \quad i = 1, 2, \ldots, n$$

(choose a basis for L_1, extend this to a basis for L_2 and so forth. Do the same for the $\tilde{L}_i$'s. Then map basis to basis. This defines the non-singular linear transformation T completely). Thus we see that $Gl(n)$ acts transitively on $Fl(\sigma_1, \ldots, \sigma_n)$. (Smoothness of the action is easy to verify.) Thus $Fl(\sigma_1, \ldots, \sigma_n)$ has the structure of a homogeneous space of $Gl(n)$ and hence has a nice manifold topology. Moreover, by just working with orthonormal bases, we could have shown that it is also a homogeneous space of $0(n)$ the orthogonal group. Since $0(n)$ is compact it follows that $Fl(\sigma_1, \ldots, \sigma_n)$ is compact. In either case, to obtain an explicit representation of the homogeneous space, one has to compute the stabilizer of a flag. For the $Gl(n)$ action, it is an easy exercise to verify that the stabilizer consists of invertible matrices of the block upper-triangular form (in a suitable basis)

$$\begin{bmatrix} A_{11} & A_{12} \ldots & A_{1n} \\ 0 & A_{22} \ldots & A_{2n} \\ \vdots & \vdots \quad \ddots & \\ 0 & 0 \quad \ldots & A_{nn} \end{bmatrix}$$

where each A_{ij} is a $\sigma_i \times \sigma_j$ matrix.

Example 6

$n=2$; $\sigma_1=1$, $\sigma_2=1$; we are dealing with the collection of lines through the origin in $\mathbb{R}^2$. Stabilizer $=\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$: invertible 2×2 upper triangular matrices. The flag manifold $Fl(1, 1)=S^1$ the circle !

We already saw how to assign a flag $(\mathscr{B}_0, \mathscr{B}_1, \dots, \mathscr{B}_{n-1})$ to a controllable pair $[A, B]$. The question is can we go backwards, i.e. given a flag can we write down a system whose flag it is by the canonical association, ((4.2), (4.3)) ? The answer is yes. We proceed in four steps.

(*a*) Given (n, m) and $\sigma=(\sigma_1, \sigma_2, \dots, \sigma_n, \sigma_1\geqslant\sigma_2\geqslant \dots \geqslant\sigma_n)$, fix the standard basis in $\mathbb{R}^n$, denoted as $\{e_1, e_2, \dots, e_{\sigma_1}, \dots, e_{\sigma_1+\rho_2}, \dots, e_n\}$.

(*b*) Let $\mathscr{V}^0=(L_1{}^0, \dots, L_n{}^0)$ be the standard flag defined by

$$L_1{}^0=\operatorname{span}\{e_1, \dots, e_{\sigma_1}\}$$
$$L_2{}^0=\operatorname{span}\{e_1, \dots, e_{\sigma_1}, \dots, e_{\sigma_1+\sigma_2}\}$$
$$\vdots$$
$$L_n{}^0=\mathbb{R}^n$$

(*c*) Consider the pair $[A_\sigma, B_\sigma]$ defined by

$$A_\sigma=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & A_{n-1} & 0 \end{bmatrix}, \quad B_\sigma=\left[\begin{array}{c|c} I_{\sigma_1} & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right].$$

where A_i is a $\sigma_{i+1}\times\sigma_i$ matrix of the form $[I'_{\sigma_{i+1}}\ 0]$ and each I_k is the $k\times k$ identity matrix.

Then the canonical flag of the pair $[A_\sigma, B_\sigma]$ is precisely $(L_1{}^0, L_2{}^0, \dots, L_n{}^0)$. Thus

$$\mathscr{V}^0=\mathscr{V}_{[A_\sigma, B_\sigma]} \tag{4.11}$$

(*d*) Now given any other flag $\mathscr{V}=(\mathscr{B}_0, \mathscr{B}_1, \dots, \mathscr{B}_{n-1})\in Fl(\sigma_1, \dots, \sigma_n)$, by the homogeneity of the flag manifold, there exists a $P\in Gl(n)$ such that

$$P\mathscr{V}^0=\mathscr{V} \tag{4.12}$$

But, from (4.5) the intertwining property

$$\begin{aligned} P\mathscr{V}^0 &= P\mathscr{V}_{[A_\sigma, B_\sigma]} \\ &= \mathscr{V}_{[PA_\sigma P^{-1}, PB_\sigma]} \end{aligned}$$

Thus $\mathscr{V}=\mathscr{V}_{[PA_\sigma P^{-1}, PB_\sigma]}$. The pair $[PA_\sigma P^{-1}, PB_\sigma]$ is the desired pair. The pair $[A_\sigma, B_\sigma]$ is a canonical form for the feedback group (Martin 1979).

Theorem 6

The map

$\pi_F : \Sigma_{n,m}{}^r(\sigma) \subseteq$

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is a surjection.

Actually the map π_F is very w
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Theorem 7

The triple $(\tilde{\mathscr{O}}_{[A,B]}, \tilde{\pi}_F, Fl(\sigma))$ d
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Caveat

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If we denote the stabilizer of $\mathscr{V}_{[A}$

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is a closed subgroup and the fibre
space.

Theorem 8

The triple $(\mathscr{O}_{[A,B]}, \pi_F, Fl(\sigma))$ is
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If $[A, B]=[A_\sigma, B_\sigma]$ is actuall
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(4.10). A similar calculation has
(19—) formulae correspond to th
See Martin (1979) for details.

Theorem 6

The map

$$\pi_F : \Sigma_{n,m}{}^r(\sigma) \subseteq \Sigma_{n,m}{}^r \to Fl(\sigma)$$
$$[A, B] \to (\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_{n-1})$$

is a surjection.

Actually the map π_F is very well behaved and is of use in further unravelling the structure of $\Sigma_{n,m}{}^r(\sigma)$. Notice that $Gl(n)$ acts on $\Sigma_{n,m}{}^r(\sigma)$ via

$$[A, B] \to [PAP^{-1}, PB]$$

and the action is free (because of controllability). So if one restricts the map π_F to an orbit $\tilde{\mathscr{O}}_{[A,B]}$ of $Gl(n)$ (which looks like a copy of $Gl(n)$) then the inverse image of $\tilde{\pi}_F = \pi_F | \tilde{\mathscr{O}}_{[A,B]}$ is diffeomorphic to the stabilizer of the flag $\mathscr{V}_{[A,B]}$. More precisely we may state the following theorem.

Theorem 7

The triple $(\tilde{\mathscr{O}}_{[A,B]}, \tilde{\pi}_F, Fl(\sigma))$ defines a fibration of $\tilde{\mathscr{O}}_{[A,B]}$ with fibres diffeomorphic to the stabilizer of the flag $\mathscr{V}_{[A,B]}$.

Caveat

It does not necessarily hold that if the systems have the same flag, they are $Gl(n)$ equivalent (i.e. they are on the same orbit $\tilde{\mathscr{O}}_{[A,B]}$). This is because the same fibre of π_F can intersect many $Gl(n)$ orbits.

Recall from the beginning of this section that the feedback group $\mathscr{F}$ itself may be viewed as acting on $Fl(\sigma)$. Since by definition

$$\mathscr{F}_{[A,B]} \cdot [A, B] = \{[A, B]\} \tag{4.13}$$

we have

$$\mathscr{F}_{[A,B]} \cdot \mathscr{V}_{[A,B]} = \{\mathscr{V}_{[A,B]}\} \tag{4.14}$$

If we denote the stabilizer of $\mathscr{V}_{[A,B]}$ in $\mathscr{F}$ as $\mathscr{K}_{[A,B]}$, then

$$\mathscr{F}_{[A,B]} \subseteq \mathscr{K}_{[A,B]} \tag{4.15}$$

is a closed subgroup and the fibre $\pi^{-1}(\mathscr{V}_{[A,B]}) \simeq \mathscr{K}_{[A,B]}/\mathscr{F}_{[A,B]}$ a homogeneous space.

Theorem 8

The triple $(\mathscr{O}_{[A,B]}, \pi_F, Fl(\sigma))$ is a fibre bundle with fibre $\simeq \mathscr{K}_{[A,B]}/\mathscr{F}_{[A,B]}$ a homogeneous space.

If $[A, B] = [A_\sigma, B_\sigma]$ is actually in canonical form, then we have already computed the group $\mathscr{K}_{[A_\sigma, B_\sigma]}$ is a group of block upper triangular form, eqn. (4.10). A similar calculation has to be done for $\mathscr{F}_{[A_\sigma, B_\sigma]}$ whereas Brockett's (19—) formulae correspond to the Brunovsky (1970) normal form $[A_\rho, B_\rho]$. See Martin (1979) for details.

Example 7

Single-input case : $m=1$; $\sigma=(\sigma_1, \ldots, \sigma_n)=(1, 1, \ldots, 1)$. Let $[A_\sigma, b_\sigma]$ be in the canonical form

$$A=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix}, \quad b=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then $Fl(\sigma)=0(n)/D$, where D is a discrete subgroup. The stabilizer for the feedback action on $[A_\sigma, b_\sigma]$ is the single dimensional group

$$\mathscr{F}_{[A_\sigma, b_\sigma]}=\{\alpha I_n : \alpha \neq 0\}$$

The stabilizer of the feedback action on $\mathscr{V}_{[A_\sigma, b_\sigma]}$ is

$$\mathscr{K}_{[A_\sigma, b_\sigma]}=\mathbb{R}^n \otimes_s \text{Ptr}\,(n)$$

where $\mathbb{R}^n$ is the additive abelian group, Ptr (n) denotes the projective upper triangular group of $(n \times n)$ matrices and $\otimes_s$ denotes semidirect product. One can now use Theorem 8 to obtain the set of all controllable single-input systems as a fibre bundle over $0(n)/D$.

We hope to have demonstrated in this section that, by a deeper study of the stabilizer and the flag manifold, etc., associated with a given orbit, the rich structure of the orbit stands revealed. The intrinsic power of the geometric viewpoint for the study of families of systems becomes apparent in the new results that we are led to that would have remained inaccessible otherwise.

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