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On the equilibria of rigid spacecraft with rotors

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We determine the structure of the set of equilibrium points for the dynamics of a rigid spacecraft carrying motor-driven rotors. In particular, our calculations make clear that it is always possible to adjust the constant angular velocities (relative to the spacecraft) of the driven rotors in such a way that there are precisely two equilibrium points. This latter situation is of interest for the purpose of ensuring satisfactory asymptotic behavior in the case of dual spin attitude acquisition maneuvers.

Keywords: Attitude control, Critical point theory, Asymptotic stability, Dual-spin spacecraft.

1. Introduction

Consider a rigid spacecraft with three motor-driven rotors (see Figure 1). By choosing the motor torques appropriately it is possible to maintain the wheels spinning at constant angular velocities relative to the spacecraft. From [3] the governing equations are

$$\dot{h}_v = S(J_a^{v-1} h_v)[h_v + h_w], \tag{1.1a}$$

$$\dot{h}_w = 0. \tag{1.1b}$$

Here for a vector $x = (x_1, x_2, x_3)'$ in R^3, $S(x)$ denotes the matrix

$$\begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}.$$

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Furthermore

$$h_v = J_a^v \omega_a, \quad h_w = J_a^w \omega_a^w,$$

J_a^v is the moment of inertia of the spacecraft together with wheels locked relative to body frame,

$$J_a^w = \mathrm{diag}(j_1^w, j_2^w, j_3^w)$$

is the matrix of rotor moments of inertia, ω_a is the spacecraft angular velocity vector, and ω_a^w is the vector of velocities of rotors relative to the spacecraft. In [3], it was shown that (1.1) is in Lie–Poisson form and hence is a Hamiltonian system when restricted to the momentum sphere (i.e. the coadjoint orbit in the dual of so(3) $\oplus R^3$),

$$S_\mu^2 = \{ h_v : \|h_v + h_w\|^2 = \mu^2 \}, \quad \mu > 0. \tag{1.2}$$

The corresponding Hamiltonian is

$$H_w = \tfrac{1}{2}\langle h_v, J_a^{v-1} h_v \rangle. \tag{1.3}$$

We are interested in the following two problems:

(P1) Determine the structure of the set of equilibrium points of (1.1).

(P2) Determine conditions on the parameter h_w such that there are precisely two equilibrium points (on the momentum sphere).

Since (1.1) is a Hamiltonian system when restricted to the momentum sphere, the equilibrium points of (1.1) are precisely the critical points of the Hamiltonian H_w restricted to the momentum sphere. Thus we can restate problem (P2) in the equivalent form:

(P2)* Determine conditions on h_w such that H_w is a perfect Morse function on the sphere S_μ^2.

We solve these two problems below.

2. A basic lemma

Let Σ_{μ,h_w} denote the set of equilibrium points of (1.1). Then,

$$\Sigma_{\mu,h_w} = \{ h_v : \|h_v + h_w\|^2 = \mu^2, \\ S(J_a^{v-1} h_v)[h_v + h_w] = 0 \}$$

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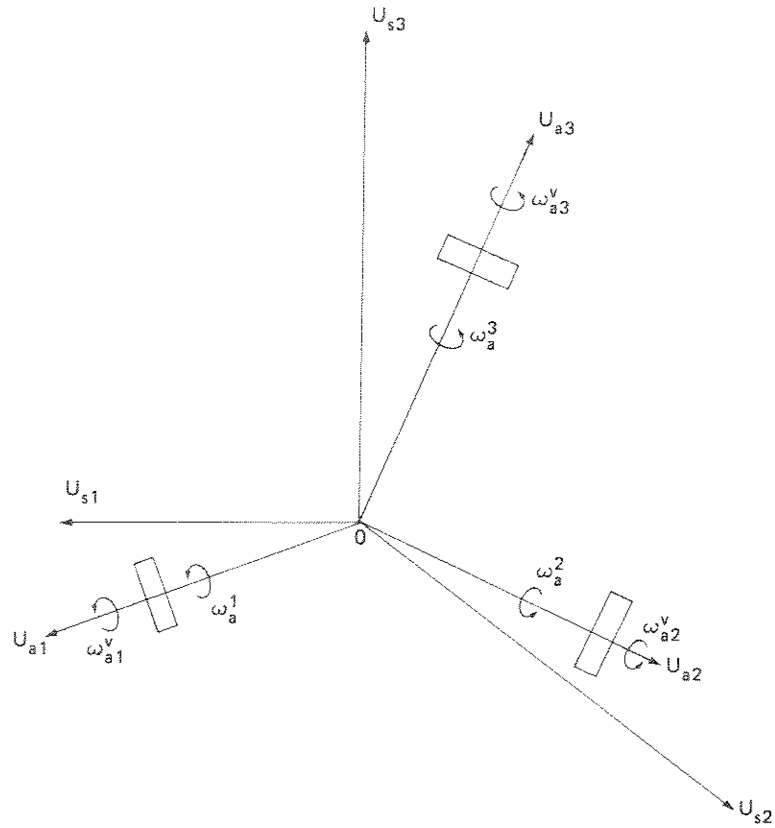

Fig. 1.

$$= \{ h_v \colon \mathrm{d} H_w^\mu(h_v) = 0, \|h_v + h_w\|^2 = \mu^2 \}. \tag{2.1}$$

In (2.1) H_w^μ denotes the restriction of H_w to the sphere S_μ^2 and $\mathrm{d} H_w^\mu$ is the associated 1-form.

Change of coordinates $h = h_v + h_w$ implies

$$S_\mu^2 = \{ h \colon \|h\|^2 = \mu^2 \}$$

and

$$\Sigma_{\mu, h_w} = \{ h \colon \|h\|^2 = \mu^2, \mathrm{d} \tilde{H}_w^\mu(h) = 0 \},$$

where

$$\tilde{H}_w(h) = \tfrac{1}{2} \langle (h - h_w), J_\mathrm{a}^{v-1}(h - h_w) \rangle.$$

If $y \in R^3$ then $y \in (\mathrm{TS}_\mu^2)_h$, the tangent space to S_μ^2 at h, if $y \perp h$. Further,

$$(\mathrm{d} \tilde{H}_w(h))(y) = \langle J_\mathrm{a}^{v-1}(h - h_w), y \rangle.$$

Thus,

$$(\mathrm{d} \tilde{H}_w^\mu(h))(y) = 0 \quad \text{for all } y \in \left(\mathrm{TS}_\mu^2\right)_h,$$

$$\Leftrightarrow \quad (\mathrm{d} \tilde{H}_w(h))(y) = 0 \quad \text{for all } y \perp h,$$

$$\Leftrightarrow \quad \langle J_\mathrm{a}^{v-1}(h - h_w), y \rangle = 0 \quad \text{for all } y \perp h,$$

$$\Leftrightarrow \quad J_\mathrm{a}^{v-1}(h - h_w) = \lambda h \quad \text{for some } \lambda \in R.$$

We have thus proved:

Lemma 2.1. *The set of equilibrium points of* (1.1) *is given by* $h_v = h - h_w$ *where* $h \in \Sigma_{\mu, h_w}$ *with*

$$\Sigma_{\mu, h_w} = \{ h \colon \|h\|^2 = \mu^2, \; \left(J_\mathrm{a}^{v-1} - \lambda I \right) h = J_\mathrm{a}^{v-1} h_w \; \text{for some } \lambda \in R \}.$$

3. Diagonalization

Since J_a^v is symmetric positive definite, let

$$M' J_\mathrm{a}^v M = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$

where $M \in O(3)$. Let

$$\tilde{h}_w = M'h_w, \quad \tilde{h} = M'h$$

and denote

$$\tilde{h}_w = (q_1, q_2, q_3)', \quad \tilde{h} = (p_1, p_2, p_3)'.$$

From Lemma 2.1,

$$\Sigma_{\mu, h_w} = \Big\{ \tilde{h}: \|\tilde{h}\|^2 = \mu^2, \\ (\Lambda^{-1} - \lambda I)\tilde{h} = \Lambda^{-1}\tilde{h}_w \\ \text{for some } \lambda \in R \Big\}.$$

Thus, given (q_1, q_2, q_3) we are led to find p_1, p_2, p_3 such that for some $\lambda \in R$

$$(\lambda_1^{-1} - \lambda)p_1 = \lambda_1^{-1}q_1, \tag{3.1a}$$

$$(\lambda_2^{-1} - \lambda)p_2 = \lambda_2^{-1}q_2, \tag{3.1b}$$

$$(\lambda_3^{-1} - \lambda)p_3 = \lambda_3^{-1}q_3, \tag{3.1c}$$

$$p_1^2 + p_2^2 + p_3^2 = \mu^2, \quad \mu > 0. \tag{3.1d}$$

Solving (3.1) involves distinguishing two main cases:

(i) asymmetric spacecraft, all λ_i distinct;

(ii) symmetric spacecraft, λ_i not all distinct.

In the asymmetric case the equilibria are points. In the symmetric case it is possible to obtain connected one-dimensional submanifolds of equilibria. In both cases perfectness conditions (for problem (P2) or (P2)*) exist. We give details of calculations for the asymmetric case only, and state the results for the symmetric case.

Remark. We are only interested in solving (3.1) when at least one of the q_i is nonzero. Otherwise the problem reduces to that of finding the equilibria of the classical Euler equations which is well understood (see Abraham and Marsden [1], pp. 360–368).

4. Asymmetric spacecraft (λ_i distinct)

We have three cases to consider;

(i) three cases corresponding to only one of the $q_i \neq 0$,

(ii) three cases corresponding to only one of the $q_i = 0$,

(iii) $q_1 \neq 0$, $q_2 \neq 0$, $q_3 \neq 0$.

Since we do not assume any particular ordering on the λ_i, we need only treat one representative case each from (i) and (ii) above and we have a reduction to three cases as below.

Case 1. $q_1 = 0$, $q_2 = 0$, $q_3 \neq 0$.
Case 1a. $\lambda \notin \{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}$. Then

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = \pm\mu.$$

Case 1b. $\lambda = \lambda_1^{-1}$. Then

$$p_2 = 0,$$

$$p_3 = \frac{\lambda_3^{-1}q_3}{(\lambda_3^{-1} - \lambda_1^{-1})} = \frac{\lambda_1 q_3}{\lambda_1 - \lambda_3},$$

$$p_1 = \pm\sqrt{\mu^2 - \left(\frac{\lambda_1 q_3}{\lambda_1 - \lambda_3}\right)^2}.$$

Case 1c. $\lambda = \lambda_2^{-1}$. Then

$$p_1 = 0,$$

$$p_3 = \frac{\lambda_3^{-1}q_3}{\lambda_3^{-1} - \lambda_2^{-1}} = \frac{\lambda_2 q_3}{\lambda_2 - \lambda_3},$$

$$p_2 = \pm\sqrt{\mu^2 - \left(\frac{\lambda_2 q_3}{\lambda_2 - \lambda_3}\right)^2}.$$

Thus in general there are six real roots. However the roots in cases 1b and 1c will be nonreal and have to be discarded, if q_3 is large enough. We give the perfectness condition:

(PC1) If $q_1 = 0$, $q_2 = 0$, $q_3 \neq 0$ then there are precisely two critical points, $p_1 = p_2 = 0$, $p_3 = \pm\mu$, iff

$$\left(\frac{q_3}{\mu}\right)^2 \geqslant \left(1 - \frac{\lambda_3}{\lambda_1}\right)^2$$

and

$$\left(\frac{q_3}{\mu}\right)^2 \geqslant \left(1 - \frac{\lambda_3}{\lambda_2}\right)^2.$$

Case 2. $q_1 = 0$, $q_2 \neq 0$, $q_3 \neq 0$.
Case 2a. $\lambda \notin \{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}$. Then

$$p_1 = 0, \quad p_2 = \frac{\lambda_2^{-1}q_2}{\lambda_2^{-1} - \lambda}, \quad p_3 = \frac{\lambda_3^{-1}q_3}{\lambda_3^{-1} - \lambda}.$$

Since $p_1^2 + p_2^2 + p_3^2 = \mu^2$,

$$\frac{q_2^2}{(1 - \lambda\lambda_2)^2} + \frac{q_3^2}{(1 - \lambda\lambda_3)^2} = \mu^2. \tag{4.1}$$

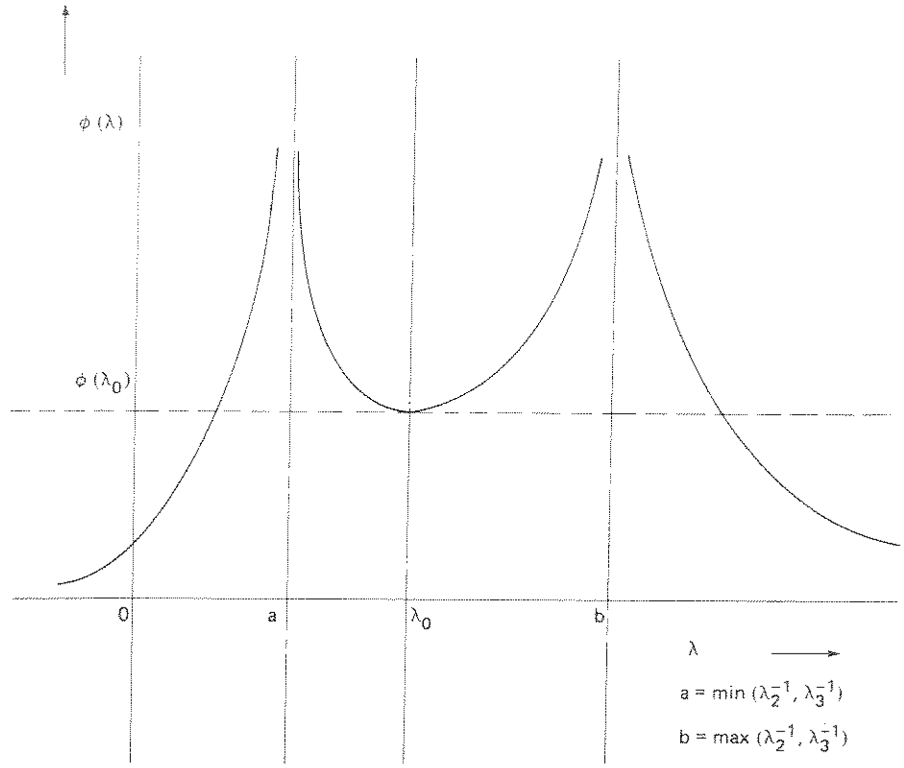

Fig. 2.

Consider the graph of the rational function

$$\phi(\lambda) = \frac{q_2^2}{(1-\lambda\lambda_2)^2} + \frac{q_3^2}{(1-\lambda\lambda_3)^2}$$

as in Figure 2. It attains its minimum value on the interval

$$\left(\min\left(\lambda_2^{-1}, \lambda_3^{-1}\right), \max\left(\lambda_2^{-1}, \lambda_3^{-1}\right)\right)$$

at λ_0 given by

$$\lambda_0 = \frac{\lambda_3\left(q_2^2\lambda_2\right)^{1/3} + \lambda_2\left(q_3^2\lambda_3\right)^{1/3}}{\left(q_2^2\lambda_2\right)^{1/3} + \left(q_3^2\lambda_3\right)^{1/3}}$$

and

$$\phi(\lambda_0) = \frac{q_3^2}{(1-\lambda_0\lambda_3)^2}\left\{1 + \frac{q_2^2}{q_3^2}\left(\frac{q_2^2}{q_3^2}\frac{\lambda_2}{\lambda_3}\right)^{2/3}\right\}.$$

Remark. We see that in the present case 2a, there are 4, 3 or 2 distinct roots depending respectively on whether $\phi(\lambda_0) < \mu^2$, $\phi(\lambda_0) = \mu^2$ or $\phi(\lambda_0) > \mu^2$. From the expressions for λ_0 and $\phi(\lambda_0)$ it should be clear that by choosing q_2 and q_3 large enough it is always possible to ensure $\phi(\lambda_0) > \mu^2$.

Case 2b. $\lambda = \lambda_1^{-1}$. Then

$$p_2 = \frac{\lambda_1 q_2}{\lambda_1 - \lambda_2}, \quad p_3 = \frac{\lambda_1 q_3}{\lambda_1 - \lambda_3},$$

$$p_1 = \pm\sqrt{\mu^2 - \left(\frac{\lambda_1 q_3}{\lambda_1 - \lambda_3}\right)^2 - \left(\frac{\lambda_1 q_2}{\lambda_1 - \lambda_2}\right)^2}.$$

We can now state the perfectness condition:

(PC2) If $q_1 = 0$, $q_2 \neq 0$, $q_3 \neq 0$, then there are precisely two critical points if

$$\mu^2 < \phi(\lambda_0)$$

and

$$\mu^2 < \left(\frac{\lambda_1 q_3}{\lambda_1 - \lambda_3}\right)^2 + \left(\frac{\lambda_1 q_2}{\lambda_1 - \lambda_2}\right)^2.$$

Case 3. $q_1 \neq 0$, $q_2 \neq 0$, $q_3 \neq 0$.

Then necessarily $\lambda \notin \{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}$,

$$p_1 = \frac{q_1}{1-\lambda\lambda_1}, \quad p_2 = \frac{q_2}{1-\lambda\lambda_2}, \quad p_3 = \frac{q_3}{1-\lambda\lambda_3},$$

and λ satisfies

$$\sum_{i=1}^{3} \frac{q_i^2}{(1-\lambda\lambda_i)^2} = \mu^2. \tag{4.2}$$

Let

$$\phi(\lambda) = \sum_{i=1}^{3} \frac{q_i^2}{(1-\lambda\lambda_i)^2}.$$

Then the graph of the function $\phi(\lambda)$ is as in Figure 3, where

$$a = \min\{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\},$$
$$c = \max\{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}$$

and b is the remaining element. λ_0^+ and λ_0^- are local minima. (Their roles may be interchanged without loss of generality.) Figure 3 clearly indicates the change in the number of real roots as the range of μ varies. We have the perfectness condition:

(PC3) If $q_1 \neq 0$, $q_2 \neq 0$, $q_3 \neq 0$, then there are precisely two critical points iff

$$\mu^2 < \min(\phi(\lambda_0^+), \phi(\lambda_0^-)).$$

The values of λ_0^+ and λ_0^- can be determined by elementary calculations and are omitted. By arguments similar to those for case 2a it can be verified that the condition (PC3) can be ensured by making q_1, q_2, q_3 large enough.

5. Symmetric spacecraft

There are essentially two possibilities:

(a) complete symmetry: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_*$,

(b) axial symmetry: $\lambda_* = \lambda_1 = \lambda_2 \neq \lambda_3$ (or its two cyclic variations).

Case (a) is the simplest. Let

$$\phi(\lambda) = \frac{\sum_{i=1}^{3}(\lambda_*^{-1} q_i)^2}{(\lambda_*^{-1} - \lambda)^2}.$$

Then the equation $\phi(\lambda) = \mu^2$ always has two real roots $\lambda_\pm$. The corresponding momenta are

$$p_i^\pm = \frac{\lambda_*^{-1} q_i}{\lambda_*^{-1} - \lambda_\pm}, \quad i = 1, 2, 3.$$

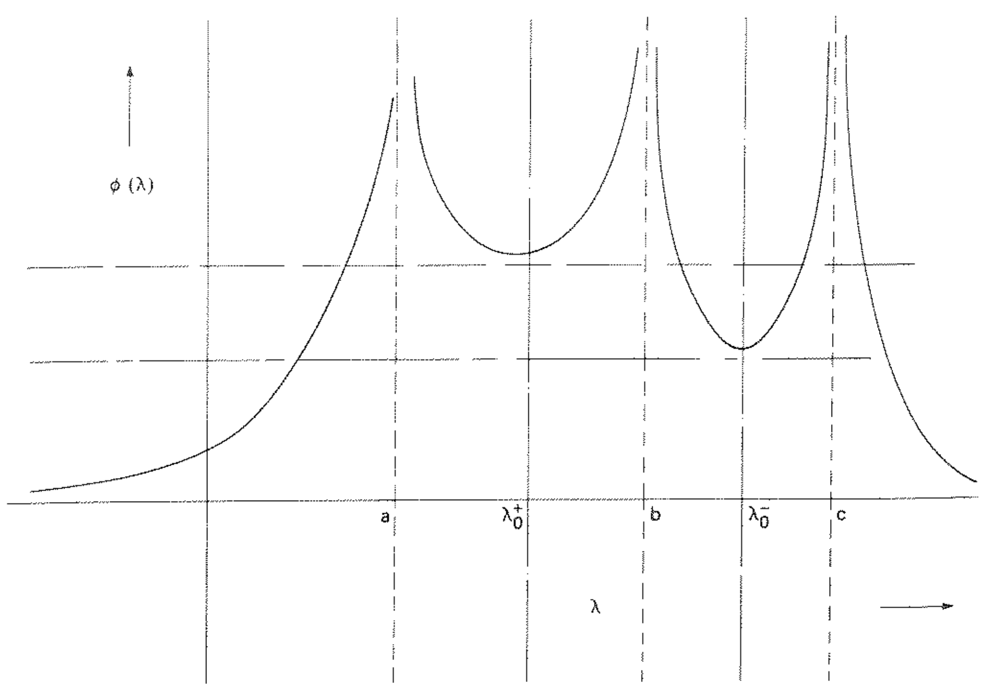

Fig. 3.

Table 1

Case	Perfectness condition
$q_1 = q_2 = 0,\ q_3 \neq 0$	$\left(\frac{q_3}{\mu}\right)^2 > \left(\frac{\lambda_* - \lambda_3}{\lambda_*}\right)^2$
$q_1 = 0,\ q_2 \neq 0,\ q_3 \neq 0$	$\mu^2 < \phi(\lambda_0)$ and $\left(\frac{q_3}{\mu}\right)^2 > \left(\frac{\lambda_* - \lambda_3}{\lambda_*}\right)^2$
	where $\phi(\lambda) = \frac{q_3^2}{(1-\lambda\lambda_3)^2} + \frac{q_2^2}{(1-\lambda\lambda_*)^2}$ and $\lambda_0 = \frac{\lambda_3(q_2^2\lambda_*)^{1/3} + \lambda_*(q_3^2\lambda_3)^{1/3}}{(q_2^2\lambda_*)^{1/3} + (q_3^2\lambda_3)^{1/3}}$
$q_1 \neq 0,\ q_2 \neq 0,\ q_3 \neq 0$	$\mu^2 < \psi(\tilde{\lambda}_0)$, $\left(\frac{q_3}{\mu}\right)^2 > \left(\frac{\lambda_* - \lambda_3}{\lambda_*}\right)^2$, and $\mu^2 < \frac{\lambda_3^2(q_1^2 + q_2^2)}{(\lambda_* - \lambda_3)^2}$
	where $\psi(\lambda) = \frac{q_3^2}{(1-\lambda\lambda_3)^2} + \frac{q_1^2 + q_2^2}{(1-\lambda\lambda_*)^2}$ and $\tilde{\lambda}_0 = \frac{\lambda_3((q_1^2+q_2^2)\lambda_*)^{1/3} + \lambda_*(q_3^2\lambda_3)^{1/3}}{((q_1^2+q_2^2)\lambda_*)^{1/3} + (q_3^2\lambda_3)^{1/3}}$

Thus perfectness is automatic (as long as one of the q_i' is nonzero).

For the case of axial symmetry assume $\lambda_1 = \lambda_2 = \lambda^*$, $\lambda_3 \neq \lambda^*$.

We summarize the perfectness conditions in Table 1.

6. Applications

For a given μ (norm of the total angular momentum of the spacecraft), it is possible to achieve the perfectness conditions of this paper by choosing q_i large enough in absolute value. This can always be accomplished by spinning up the rotors to high enough angular velocities relative to the spacecraft. This maneuver will not alter μ, since there are no external torques. When the perfectness conditions hold the resulting equilibria form a maximum–minimum pair for the Hamiltonian, and hence one of these is a stable equilibrium.

For obtaining satisfactory asymptotic behavior, it is necessary to introduce damping. This can be done by providing an additional set of 3 free-spinning rotors with damping. In this case the governing equations are [3]

$$\dot{h}_v = S(J^{-1}h_v)[h_v + h_w + h_d] - \gamma h_v + \delta h_d, \tag{6.1a}$$

$$\dot{h}_d = \gamma h_v - \delta h_d, \tag{6.1b}$$

$$\dot{h}_w = 0, \tag{6.1c}$$

where

$$h_d = J_a^d(\omega_a + \omega_a^d), \quad h_w = J_a^w \omega_a^w,$$

$$h_v = J w_a, \quad J = J_a^v - J_a^d,$$

J_a^v is the spacecraft moment of inertia with all rotors locked,

$$J_a^d = \mathrm{diag}(j_1^d, j_2^d, j_3^d)$$

is the matrix of damper moments of inertia,

$$J_a^w = \mathrm{diag}(j_1^w, j_2^w, j_3^w)$$

is the matrix of driven rotor moments of inertia, ω_a is the vector of spacecraft angular velocities, ω_a^w is the vector of driven rotor angle velocities (relative to platform), ω_a^d is the vector of damper angular velocities (relative to platform), $\gamma = \alpha J^{-1}$, $\delta = \alpha J_a^{d-1}$, and $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ is the matrix of positive damping coefficients. It can be verified (see [3]) that the equilibrium points of (6.1) are given by (for fixed h_w)

$$\begin{aligned} \Sigma_{\mu,h_w}^d = \{(h_v, h_d) : {}& h_v = J J_a^{v-1} h, \\ & h_d = J_a^d J_a^{v-1} h,\ S(J_a^{v-1}h)(h + h_w) = 0, \\ & \|h + h_w\|^2 = \mu^2\}. \end{aligned} \tag{6.2}$$

It follows from (6.2) that the perfectness conditions of this paper are also the conditions for Σ_{μ,h_w}^d to have just two isolated points only. An application of LaSalle's theorem then guarantees convergence to the stable equilibrium for a dense set of trajectories of (6.1). See [3] and its sequel [4] for details.

Notes. (1) The perfectness condition (PC1) appears in the aerospace literature in a slightly different form (see [2]). Our results give a complete solution to both the three rotors problems and to the case of nonprincipal axes. Also our basic lemma sets up our problem in the correct Hamiltonian framework.

(2) We remark that implicit in our model (1.1) is the assumption that each of the rotors is perfectly symmetrical with spin axis passing through the center of mass. In the absence of such symmetry, the dynamics become much more complex (see [7]).

(3) The reader will find related results in Crouch [6] (especially pp. 8–12).

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