Coordinated Orbit Transfer for Satellite Clusters

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ABSTRACT: We propose a control law that allows a satellite formation to achieve orbit transfer. During the transfer, the formation can be either maintained or modified to a desired formation. Based on the orbit transfer control law proposed by Chang, Chichka, and Marsden for single satellite, we add coupling terms to the summation of Lyapunov functions for single satellites. These terms are functions of the difference between the mean anomalies (or perigee passage times) of formation members. The asymptotic stability of the desired formation in desired orbits is proved.

KEYWORDS: momentum map; control Lyapunov function; formation control; orbit transfer

INTRODUCTION

This paper is concerned with the problem of achieving a satellite formation near a designated elliptic orbit. For orbits near the earth, one can use a space shuttle to place satellites into specified relative positions. What we want to consider here is the case when members of a cluster have been placed relatively far apart. They have to use their on-board thrusters to get to the desired orbits and achieve the desired formation. A similar case is when the entire formation needs to be restructured for mission-related reasons. We want the formation to be maintained to some extent during the transfer and be reestablished after the transfer.

Our approach is to use Lyapunov functions to design the control laws for orbit transfer. The Lyapunov function achieves a local minimum when the correct orbit and formation are reached. In Reference 1, a Lyapunov function is expressed as a quadratic function of the differences of orbital elements between current orbit and the destination orbit. However, in that work convergence of the associated control law is not proved.

In this paper, we develop new control laws for orbit transfer of formations and give a proof of convergence. This algorithm is based on a Lyapunov function on the shape space of elliptic orbits in our previous work,[2] the work of Chang, Chichka, and Marsden,[3] and a result of Cushman and Bates.[4] However, the most significant extension is the addition of a coupling term that is a function of the difference between

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the osculating perigee passage times. For circular orbits, we use the ascending node passage time instead of the perigee passage time.

In the next section, we develop formulæ used in the proofs of our theorems. Thereafter, we give a brief summary of results published elsewhere.[2,3] are given.We then introduce the definition of periodic satellite formations, followed by our main results and proofs about orbit transfer of periodic formations. Simulation results are provided in the final section of our paper.

CONTROL OSCULATING ELEMENTS

If the mass of a satellite is small compared to the mass of the Earth, the Kepler two body problem can be approximated by a one center problem as follows:

$$m\ddot{q} = -\nabla V_G + u, \tag{1}$$

where $q \in \mathbb{R}^3$ is the position vector of the satellite relative to the center of the Earth, m is the mass of the satellite, V_G is the gravitational potential of the Earth, u is the control force plus other disturbances. Without considering higher order terms, V_G takes the form

$$V_G = -m\frac{\mu}{\|q\|}. \tag{2}$$

Let $p = m\dot{q}$ be the momentum vector of the satellite. For simplicity we assume that all the satellites considered in this paper have unit mass.

We define the following:

$$
\begin{aligned}
l(t) &= q(t) \times p(t) \\
A(t) &= p(t) \times l(t) - \mu\frac{q(t)}{\|q(t)\|} \\
e(t) &= \frac{\|A(t)\|}{\mu} \\
a(t) &= \frac{h(t)^2}{\mu(1 - e(t)^2)} \\
\cos(E(t)) &= \frac{1}{e(t)}\left(1 - \frac{r(t)}{a(t)}\right) \\
M(t) &= E(t) - e(t)\sin(E(t)),
\end{aligned}
\tag{3}
$$

where $h(t) = \|l(t)\|$ and $r(t) = \|q(t)\|$ (these formulæ can be found in textbooks on celestial mechanics[5]); l is the angular momentum vector and A is called the Laplace vector (both are conserved if $u(t) \equiv 0$); a is the length of the semimajor axis; e is the eccentricity; E is the eccentric anomaly; and M is the mean anomaly. The last equation is Kepler's equation. When $e(t) = 0$, the eccentric anomaly $E(t)$ is defined to be $M(t)$, which is measured from the line of ascending node. For now, we assume that $e(t) \neq 0$.

Note that these formulæ are valid for all t and all the elements are differentiable on $\mathbb{R}^3 \times \mathbb{R}^3 - \{0\}$. Hence, we can take derivative on both sides of Equations (3). By using the property that l, A, a, and e are conserved when $u(t) \equiv 0$, we have

$$\dot{l}(t) = \frac{\partial l}{\partial q}\dot{q} + \frac{\partial l}{\partial p}\dot{p}$$

$$= \frac{\partial l}{\partial q}\dot{q} + \frac{\partial l}{\partial p}(-\nabla V_G + u) \tag{4}$$

$$= \dot{l}\big|_{u=0} + \frac{\partial l}{\partial p}u(t)$$

$$= \frac{\partial l}{\partial p}u(t).$$

Note that we used the property that l is conserved when $u(t) \equiv 0$. Similarly,

$$\dot{A}(t) = \frac{\partial A}{\partial p}u(t). \tag{5}$$

We obtain the derivatives of the orbital elements in the APPENDIX, the results are listed below

$$\dot{e}(t) = \frac{1}{\mu}\left(\frac{\partial A}{\partial p}\right)^{T}\hat{A} \cdot u(t), \tag{6}$$

where $\hat{A}$ is the unit vector along the direction of A. We also have

$$\dot{a}(t) = \frac{2}{\mu(1-e^2)}\left(\frac{\partial l}{\partial p}\right)^{T}l \cdot u(t) + \frac{2ae}{\mu(1-e^2)}\left(\frac{\partial A}{\partial p}\right)^{T}\hat{A} \cdot u(t) \tag{7}$$

$$\dot{E} = \sqrt{\frac{\mu}{a}}\frac{1}{r} + \frac{\cos(E)}{e\sin(E)}\dot{e} - \frac{r}{a^2 e\sin(E)}\dot{a} \tag{8}$$

$$\dot{M} = n + \eta\left(\frac{\partial A}{\partial p}\right)^{T}\hat{A} \cdot u(t) + \xi\left(\frac{\partial l}{\partial p}\right)^{T}l \cdot u(t), \tag{9}$$

where $n = \sqrt{\mu/a^3}$ and

$$\xi(l, A, E) = -\frac{2(1 - e\cos(E))^2}{\mu ae(1-e^2)\sin(E)}$$

$$\eta(l, A, E) = \frac{\cos((E)-e)}{\mu e\sin(E)} - \frac{2(1-e\cos(E))^2}{\mu(1-e^2)\sin E}. \tag{10}$$

Note that ξ and η are infinite if $\sin(E) = 0$. Physically, this means that when the satellite is passing perigee or apogee, the control can cause a sudden jump of the mean and eccentric anomalies. To prevent this from happening in our control laws, we turn off the control when $\sin(E) = 0$.

ORBIT TRANSFER OF SINGLE SATELLITE

For a single satellite on an elliptic orbit, the set D of ordered pairs (l, A) is a subset of $\mathbb{R}^3 \times \mathbb{R}^3$ with Euclidean norm,

$$D = \{(l, A) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, A \cdot l = 0, l \neq 0, \|A\| < m^2\mu\}. \tag{11}$$

Let $W = \frac{1}{2}m\|\dot{q}\|^2 + V_G$ be the total energy of the satellite. Let P be the set of all pairs (q, p) with Euclidean norm. Then, we define a set Σ_e by

$$\Sigma_e = \{(q, p) \in P \,|\, W(q, p) < 0, l \neq 0\}. \tag{12}$$

By definition of l and A, we have a mapping $\pi : \Sigma_e \to D$, $(q,p) \mapsto (l,A)$.

Theorem 1: (Chang–Chichka–Marsden)[3]
The following hold:
1. Σ_e is the union of all elliptic Keplerian orbits.
2. $\pi(\Sigma_e) = D$ and $\Sigma_e = \pi^{-1}(D)$.
3. The fiber $\pi^{-1}(l,A)$ is a unique (oriented) elliptic Keplerian orbit for each $(l,A) \in D$ (see also Ref. 4, page 58).

The mapping π is a continuous mapping because (l,A) are continuous with respect to (q,p).

Corollary 2:
$\pi^{-1}(K)$ is compact for any compact set $K \subset D$ (c.f., Ref. 2).

If we have $n \geq 1$ satellites, let $\overline{D} = D_1 \times D_2 \times \ldots \times D_n$ and let

$$\overline{d} = d_1\big|_{D_1} + d_2\big|_{D_2} + \ldots + d_n\big|_{D_n}.$$

We define the *shape space of elliptic Keplerian orbits* to be the set $\overline{D}$ with the distance function $\overline{d}$. The above theorem and corollary can be extended to $(\overline{D}, \overline{d})$.

To control the orbit transfer of a single satellite, one considers a Lyapunov function[3]

$$V(q, p) = \frac{1}{2}(\|l - l_d\|^2 + \|A - A_d\|^2), \tag{13}$$

where (l_d, A_d) is the pair of the angular momentum vector and Laplace vector of the target elliptic (circular) orbit. The derivative of V along the integral curve of the system is

$$
\begin{aligned}
\dot{V} &= (l - l_d) \cdot \dot{l} + (A - A_d) \cdot \dot{A} \\
&= (l - l_d) \cdot \frac{\partial l}{\partial p}u + (A - A_d) \cdot \frac{\partial A}{\partial p}u \\
&= [(l - l_d) \times q + l \times (A - A_d) + (A - A_d) \times p \times q] \cdot u.
\end{aligned} \tag{14}
$$

If we let the control be

$$u = -[(l - l_d) \times q + l \times (A - A_d) + ((A - A_d) \times p) \times q], \tag{15}$$

then $\dot{V} \leq 0$ along the trajectory of the closed loop system. The following theorem is proved.

Theorem 3: (Chang–Chichka–Marsden)
There exists $c > 0$ such that if $V(q_0, p_0) \leq c$, by applying the control law in Equation **(15)**, the trajectory of the closed loop system starting at (q_0, p_0) will asymptotically converge to the target orbit $\pi^{-1}(l_d, A_d)$.

PERIODIC FORMATION

Suppose we have a formation consisting of m satellites. Let O_j denote the orbit of the jth satellite. The orbit O_j and the position of the satellite on O_j can be described either by the six orbital elements $(a_j, e_j, I_j, \Omega_j, \omega_j, \tau_j)$ or by (l_j, A_j, τ_j) where a denotes the length of semimajor axis, e is the eccentricity, I is the inclination, Ω is the longitude of the ascending node, ω is the argument of the perigee, and τ is the perigee passage time. If O_j is elliptic or circular, except for singular cases for which some of the orbital elements are not well defined, Theorem 1 tells us that the two expressions are equivalent. Among all possible formations, we are interested in formations with periodic shape changes. We introduce the following definition.

Definition 1:
A formation is periodic when a_j, the length of semimajor axis of orbit O_j, satisfies $a_j = a > 0$ for all $j = 1, 2, \ldots, m$.

This definition is valid since all the satellites in a periodic formation have the same orbital period

$$T = \frac{2\pi\sqrt{a^3}}{\sqrt{\mu}}. \tag{16}$$

Thus, although the shape of the formation is varying, it is varying periodically.

On the other hand, given a set of orbits with identical length semimajor axis, there are infinitely many possible periodic formations. We need to specify their relative positions on the orbits to determine a specific formation. The difficulty is that the relative positions are complicated functions of time. However, the differences between the perigee passage times, $(\tau_i - \tau_j)$, are constants. Because the mean anomaly is

$$M_i = n_i(t - \tau_i), \tag{17}$$

where $n_i = 2\pi/T$, then $(M_i - M_j)$ are constants. By specifying the values of $(\tau_i - \tau_j)$ or $(M_i - M_j)$ for all i and j, a periodic formation can be uniquely determined.

TRANSFER OF SATELLITE FORMATIONS BETWEEN ELLIPTIC ORBITS

With all the tools we need developed, we are now ready to determine our control laws. To illustrate our ideas, we show the calculations for two satellites. A control law for more than two satellites can be developed based on the two-satellite case in a natural way.

To set up a periodic formation of two satellites, one can control each satellite separately to transfer to its target orbit. However, this will not assure the correct values for $(\tau_i - \tau_j)$ or $(M_i - M_j)$. To do that, additional terms involving $(\tau_i - \tau_j)$ or $(M_i - M_j)$ should be added in the summation of the Lyapunov functions for single satellites. This extension results in a cooperative orbit transfer of multiple satellites.

We introduce a variable Υ_i, defined by

$$\Upsilon_i = \begin{cases} \left(\dfrac{a_i}{a_d}\right)^{3/2} M_i, & \text{if } E_i \in (-\varepsilon, \pi + \varepsilon) \\[2em] \left(\dfrac{a_i}{a_d}\right)^{3/2} (2\pi - M_i), & \text{if } E_i \in (\pi - \varepsilon, 2\pi + \varepsilon) \end{cases} \tag{18}$$

for $i = 1, 2$, where a_d is the common length of the semimajor axes for the destination orbits.

Here, the trouble of using different expressions for the cases $E_i \in (-\varepsilon, \pi + \varepsilon)$ and $E_i \in (\pi - \varepsilon, 2\pi + \varepsilon)$ is caused by the fact that $E_i \in S^1$ a circle. Two coordinate charts are required on S^1. Here we pick the charts to be

$$\psi_1 : (-\varepsilon, \pi + \varepsilon) \to (-\varepsilon, \pi + \varepsilon) \text{ s.t. } E_i \mapsto E_i$$
$$\psi_2 : (\pi - \varepsilon, 2\pi + \varepsilon) \to (-\varepsilon, \pi + \varepsilon) \text{ s.t. } E_i \mapsto (2\pi - E_i). \tag{19}$$

The value of ε is chosen so that the two satellites are in the same chart. Because in a satellite formation the angular separations between satellites are usually small, the value of ε is small.

For $E_i \in (-\varepsilon, \pi + \varepsilon)$, we have

$$\dot{\Upsilon}_i = \left(\frac{a_i}{a_d}\right)^{3/2} \dot{M}_i + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \dot{a}_i M_i$$
$$= \sqrt{\frac{\mu}{a_d^3}} + \rho(l_i, A_i, E_i) \left(\frac{\partial A_i}{\partial p_i}\right)^T \hat{A}_i \cdot u_i(t) + \zeta(l_i, A_i, E_i) \left(\frac{\partial l_i}{\partial p_i}\right)^T l_i \cdot u_i(t), \tag{20}$$

where

$$\zeta = \left(\frac{a_i}{a_d}\right)^{3/2} \xi(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2}{\mu(1 - e_i^2)} M_i$$
$$\rho = \left(\frac{a_i}{a_d}\right)^{3/2} \eta(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2a_i e_i}{\mu(1 - e_i^2)} M_i. \tag{21}$$

For $E_i \in (\pi - \varepsilon, 2\pi + \varepsilon)$,

$$\dot{\Upsilon}_i = -\left(\frac{a_i}{a_d}\right)^{3/2} \dot{M}_i + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \dot{a}_i (2\pi - M_i)$$
$$= -\sqrt{\frac{\mu}{a_d^3}} + \rho(l_i, A_i, E_i) \left(\frac{\partial A_i}{\partial p_i}\right)^T \hat{A}_i \cdot u_i(t) + \zeta(l_i, A_i, E_i) \left(\frac{\partial l_i}{\partial p_i}\right)^T l_i \cdot u_i(t), \tag{22}$$

where

$$\zeta = -\left(\frac{a_i}{a_d}\right)^{3/2} \xi(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2}{\mu(1 - e_i^2)} (2\pi - M_i)$$
$$\rho = -\left(\frac{a_i}{a_d}\right)^{3/2} \eta(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2a_i e_i}{\mu(1 - e_i^2)} (2\pi - M_i). \tag{23}$$

Note that we have terms that explicitly contain M_i. If not handled well, these terms cause discontinuity in our control algorithm when the satellites enter a new chart. The reason for us to pick the particular charts (ψ_1, ψ_2) is to reduce the discontinuities in the derivatives of Υ_i caused by changing charts.

We design a Lyapunov function on the phase space of the two satellites. This single function has different expressions in different charts. The Lyapunov function is

$$
V = \begin{cases} V_1 + V_2 + 4\sin\left(\dfrac{\Upsilon_1 - \Upsilon_2 - \phi}{4}\right)^2, & \text{if } E_i \in (-\varepsilon, \pi + \varepsilon) \\[2mm] V_1 + V_2 + 4\sin\left(\dfrac{\Upsilon_1 - \Upsilon_2 + \phi}{4}\right)^2, & \text{if } E_i \in (\pi - \varepsilon, 2\pi + \varepsilon), \end{cases}
\tag{24}
$$

where

$$
\begin{aligned}
V_1 &= \frac{1}{2}(\|l_1 - l_{d_1}\|^2 + \|A_1 - A_{d_1}\|^2) \\
V_2 &= \frac{1}{2}(\|l_2 - l_{d_2}\|^2 + \|A_2 - A_{d_2}\|^2).
\end{aligned}
\tag{25}
$$

Here, (l_{d_1}, A_{d_1}) and (l_{d_2}, A_{d_2}) specify the orbits in a two-satellite periodic formation and ϕ specifies the desired $(M_1 - M_2)$ on these orbits.

We calculate the derivative of V as

$$
\dot{V} = \dot{V}_1 + \dot{V}_2 + \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)(\dot{\Upsilon}_1 - \dot{\Upsilon}_2).
\tag{26}
$$

The choice of $-$ or $+$ depends on the value of E_i as in the definition of V. From calculations performed in the single satellite case,

$$
\dot{V}_i = [(l_i - l_{d_i}) \times q_i + l_i \times (A_i - A_{d_i}) + ((A_i - A_{d_i}) \times p_i) \times q_i] \cdot u_i,
\tag{27}
$$

for $i = 1, 2$. Thus,

$$
\begin{aligned}
\dot{V} = & \left[\left(l_1 - l_{d_1} + \zeta_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)l_1\right) \times q_1 \right. \\[2mm]
& + l_1 \times \left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right) \\[2mm]
& + \left.\left(\left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right) \times p_1\right) \times q_1\right] \cdot u_1 \\[2mm]
& + \left[\left(l_2 - l_{d_2} + \zeta_2 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)l_2\right) \times q_2 \right. \\[2mm]
& + l_2 \times \left(A_2 - A_{d_2} + \rho_2 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_2\right) \\[2mm]
& + \left.\left(\left(A_2 - A_{d_2} + \rho_2 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_2\right) \times p_2\right) \times q_2\right] \cdot u_2.
\end{aligned}
\tag{28}
$$

To obtain $\dot{V} \leq 0$, we let

$$u_1 = -\sin^2(E_1)\left[\left(l_1 - l_{d_1} + \zeta_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)l_1\right) \times q_1\right.$$

$$+ l_1 \times \left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right)$$

$$+ \left.\left(\left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right) \times p_1\right) \times q_1\right]$$

$$u_2 = -\sin^2(E_2)\left[\left(l_2 - l_{d_2} + \zeta_2 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)l_2\right) \times q_2\right.$$

$$+ l_2 \times \left(A_2 - A_{d_2} + \rho_2 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_2\right)$$

$$+ \left.\left(\left(A_2 - A_{d_2} + \rho_2 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_2\right) \times p_2\right) \times q_2\right]. \tag{29}$$

Note that the factors $\sin^2(E_i)$ cancel the term $\sin(E_i)$ in the denominators of ζ_i and ρ_i. This results in a continuous control law that will be 0 when $E_i = 0, \pi$.

Another issue is that when $e_i = 0$, E_i is not well defined. Thus, we should not allow $A_{d_i} = 0$. Furthermore, we should choose our initial value of function V such that the subset of the space $\Sigma_{e_1} \times \Sigma_{e_2}$, where $A_i = 0$, will not be reached by the controlled dynamics.

Let $z = (q_1, p_1, q_2, p_2)$. We now proceed to find the initial condition

$$z_0 = (q_1(0), p_1(0), q_2(0), p_2(0))$$

for z, such that the set

$$S_M = \{z \,|\, V(z) \le V(z_0)\} \tag{30}$$

is a compact subset of $\Sigma_{e_1} \times \Sigma_{e_2} - \{z \,|\, A_1 = 0 \text{ or } A_2 = 0\}$. This is a necessary step because we want to apply LaSalle's invariance principle to prove our main result.

Lemma 4:
Let

$$c < \min\{c_1, c_2\}, \tag{31}$$

where

$$c_i = \min\left\{\frac{1}{2}\|A_{d_i}\|^2, \frac{1}{2}\|l_{d_i}\|^2, \frac{1}{2}(\mu - \|A_{d_i}\|)^2\right\} \tag{32}$$

for $i = 1, 2$. Then the set

$$S_M = \{z \,|\, V(z) \le c\} \tag{33}$$

is a compact subset of $\Sigma_{e_1} \times \Sigma_{e_2} - \{z \,|\, A_1 = 0 \text{ or } A_2 = 0\}$.

Proof:
The first observation is that the set

$$S_1 = \{(q_1, p_1) \,|\, V(q_1, p_1) \le c^*, c^* < c_1\} \tag{34}$$

is a subset of Σ_{e_1}, that is, $S_1 \cap \Sigma_{e_1} = S_1$. In fact, c_1 is the supremum of $V_1(q_1, p_1)$ on the set $\Sigma_{e_1} - \{(q_1, p_1)|A_1 = 0\}$. To see this, we solve a constrained maximization problem as follows:

$$\sup\left\{\frac{1}{2}\left\|l_1 - l_{d_1}\right\|^2 + \left\|A_1 - A_{d_1}\right\|^2\right\} \tag{35}$$

under the constraints

$$A_1 \cdot l_1 = 0, \ l_1 \neq 0, \ A_1 \neq 0, \ \text{and} \ \|A_1\| < \mu. \tag{36}$$

First, we need to calculate the supremum of the unconstrained maximization problem. It is easy to see that this value is infinite. Then, we need to calculate the minimum value subject to $l_1 = 0$. The result is $\frac{1}{2}\|l_d\|^2$, achieved when $A = A_d$. Similarly, the minimum subject to $A_1 = 0$ is $\frac{1}{2}\|A_d\|^2$. We should also calculate the minimum value subject to $\|A\| = \mu$. By applying the Lagrange multiplier method, we found this value to be $\frac{1}{2}(\mu - \|A_d\|)^2$. Thus,

$$c_1 = \min\left\{\frac{1}{2}\left\|A_{d_1}\right\|^2, \frac{1}{2}\left\|l_{d_1}\right\|^2, \frac{1}{2}(\mu - \left\|A_{d_1}\right\|)^2\right\} \tag{37}$$

is the supremum of V_1 on the set $\Sigma_{e_1} - \{(q_1, p_1)|A_1 = 0\}$. Thus, we have proved that $S_1 \subset \Sigma_{e_1} - \{(q_1, p_1)|A_1 = 0\}$.

Another observation is that $\pi(S_1)$ is a compact subset of D_1. Thus, by Corollary 2, S_1 is a compact subset of Σ_{e_1}.

We can make the same arguments for the case when $i = 2$ to prove that S_2 is a compact subset of $\Sigma_{e_2} - \{(q_2, p_2)|A_2 = 0\}$. Hence, by letting $c < \min\{c_1, c_2\}$, it is true that

$$S_M \subset S_1 \times S_2 \subset \Sigma_{e_1} \times \Sigma_{e_2} - \{z|A_1 = 0 \text{ or } A_2 = 0\}. \tag{38}$$

Thus, S_M is a compact subset of $\Sigma_{e_1} \times \Sigma_{e_2} - \{z|A_1 = 0 \text{ or } A_2 = 0\}$. ∎

We can now apply LaSalle's invariance principle to show that the trajectory of the closed loop system, starting within S_M, converges to the maximal invariant subset of S_M, where $u(t) = 0$ is satisfied for all t.

Proposition 5:
With V, c, and u_i given, as in **(24)**, **(31)**, and **(29)**, the trajectory starting from the point $(q_{10}, p_{10}, q_{20}, p_{20})$ that satisfies

$$V(q_{10}, p_{10}, q_{20}, p_{20}) \leq c \tag{39}$$

will converge to the set for which

$$l_i = l_{d_i}$$
$$A_i = A_{d_i} \tag{40}$$
$$M_1 - M_2 = \phi$$

are satisfied for $i = 1, 2$.

Proof:
To calculate the invariant set, let $u_1 = 0$. When $\sin(E_1) = 0$, we obtain

$$\left(l_1 - l_{d_1} + \zeta_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)l_1\right) \times q_1$$

$$+ l_1 \times \left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right) \tag{41}$$

$$+ \left(\left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right) \times p_1\right) \times q_1$$

$$= 0.$$

By taking inner products on both sides with $q_1(t)$ we obtain

$$\left(l_1 \times \left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right)\right) \cdot q_1 = 0. \tag{42}$$

This is equivalent to

$$(l_1 \times q_1(t)) \cdot \left(A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1\right) = 0. \tag{43}$$

Let

$$B = A_1 - A_{d_1} + \rho_1 \sin\left(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2}\right)\hat{A}_1, \tag{44}$$

then Equation (43) means that B is perpendicular to the vector $l_1 \times q_1(t)$. We see that the vector B should stay in the plane spanned by l_1 and $q_1(t)$, as shown in FIGURE 1. However, from the assumption that $u_1(t) = 0$, we know that A_1 and l_1 are constant vectors. The time varying vector B will sweep a line segment passing the fixed point $(A_1 - A_{d_1})$. The direction of this line segment is aligned with $\hat{A}_1$. Thus, vector B will be the intersection of the (l_1, q_1) plane and this line segment. Because $q_1(t)$ is sweeping the orbital plane, the (l_1, q_1) plane is identical at t and $t + kT_1$, where k is an integer and T_1 is the period of the first satellite. Since the line segment is not changed, the intersection points in these cases must be identical. Thus, we must have
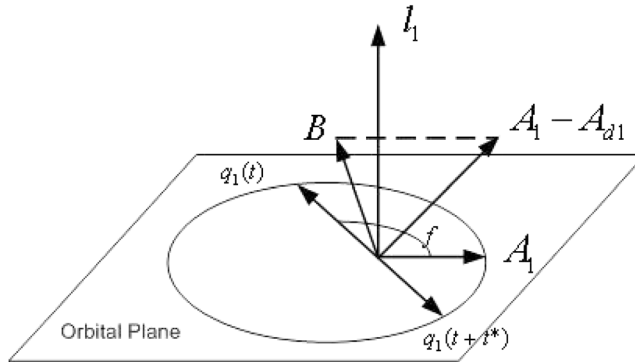
$$B(t) = B(t + kT_1). \tag{45}$$

FIGURE 1. The relationship between l_1, q_1, $A_1 - A_{d_1}$, and B (see Proposition 5).

Without lost of generality, suppose at time t, $E_1(t), E_2(t) \in (-\varepsilon, \pi + \varepsilon)$. Then, **(45)** requires that

$$\rho_1(E_1(t)) \sin\left(\frac{\Upsilon_1(t) - \Upsilon_2(t) - \phi}{2}\right)$$
$$= \rho_1(E_1(t + kT_1)) \sin\left(\frac{\Upsilon_1(t + kT_1) - \Upsilon_2(t + kT_1) - \phi}{2}\right). \tag{46}$$

Let $k = 1$ in Equation **(46)**, because $\rho_1(E_1(t)) = \rho_1(E_1(t + T_1))$, the first observation we make is that the two satellites must have the same period. In fact, suppose that at time t_0 Equation **(46)** is satisfied. Then, at time $t_0 + T_1$, since $E_1(t_0) = E_1(t_0 + T_1)$, $\Upsilon_1(t_0) = \Upsilon_1(t_0 + T_1)$, and all the angles (anomalies) are in the interval $[0, 2\pi)$, we must have $\Upsilon_2(t_0) = \Upsilon_2(t_0 + T_1)$. However,

$$\Upsilon_2(t_0) - \Upsilon_2(t_0 + T_1) = -\left(\frac{a_2}{a_d}\right)^{3/2}(M_2(t_0 + T_1) - M_2(t_0)). \tag{47}$$

Then $\Upsilon_2(t_0) = \Upsilon_2(t_0 + T_1)$ is satisfied only if $M_2(t_0 + T_1) = M_2(t_0)$. Thus, we have $T_1 = k_1 T_2$, where k_1 is a positive integer. Remember, we can apply the same argument to the second satellite to obtain $T_2 = k_1 T_1$. Thus, we must have $T_1 = T_2$. Hence, on the invariant set, we have proved that $a_1 = a_2$.

On the other hand, for a specific time t, we know that there exists $t^* \in [0, T_1)$ such that

$$\pi + f_1(t) = f_1(t + t^*), \tag{48}$$

where f_1 is the true anomaly of the first satellite. The value of t^* depends on t. The plane spanned by (l_1, q_1) at time t is also identical to the plane spanned by (l_1, q_1) at time $t + t^*$. Thus, we must have

$$B(t) = B(t + t^*), \tag{49}$$

which requires that

$$\rho_1(E_1(t)) \sin\left(\frac{\Upsilon_1(t) - \Upsilon_2(t) - \phi}{2}\right)$$
$$= \rho_1(E_1(t + t^*)) \sin\left(\frac{\Upsilon_1(t + t^*) - \Upsilon_2(t + t^*) + \phi}{2}\right). \tag{50}$$

Furthermore, $a_1 = a_2$ implies that $M_1(t) - M_2(t) = M_1(t + t^*) - M_2(t + t^*)$, and one can verify that

$$\sin\left(\frac{\Upsilon_1(t) - \Upsilon_2(t) - \phi}{2}\right) = -\sin\left(\frac{\Upsilon_1(t + t^*) - \Upsilon_2(t + t^*) + \phi}{2}\right). \tag{51}$$

For **(50)** to be satisfied, one possibility is that

$$\sin\left(\frac{\Upsilon_1(t) - \Upsilon_2(t) - \phi}{2}\right) = 0. \tag{52}$$

Another possibility is that

$$\rho_1(E_1(t)) = -\rho_1(E_1(t + t^*)). \tag{53}$$

By definition of ρ_1, one can verify that **(53)** can only be satisfied when t takes value from a set of measure 0. Thus, for **(50)** to be satisfied, **(52)** must be true. Because of

(52), the time varying parts in Equation **(41)** vanish. We can make the same argument as in the proof for the single satellite case[3] to show that

$$l_1 = l_{d_1}$$
$$A_1 = A_{d_1}.$$

(54)

We can apply similar arguments for the second satellite. Thus, we have

$$l_i = l_{d_i}$$
$$A_i = A_{d_i}$$
$$\Upsilon_1 - \Upsilon_2 = \pm \phi$$

(55)

for $i = 1, 2$. By definition of Υ_1 and Υ_2 in Equation **(18)**, we have

$$\left(\frac{a_1}{a_d}\right)^{3/2} M_1 - \left(\frac{a_2}{a_d}\right)^{3/2} M_2 = \phi.$$

(56)

However, we already know $a_1 = a_2 = a_d$, so we conclude that

$$M_1 - M_2 = \phi.$$

(57)

■

This proposition can be easily extended to m satellites where $m > 2$. Let the mth satellite be the leader of the formation. Let

$$V = \sum_i^m \left(V_i + 4\sin\left(\frac{\Upsilon_1 - \Upsilon_m \mp \phi_i}{4}\right)^2\right),$$

(58)

where $i = 1, 2, 3, \ldots, m$, $\phi_m = 0$, and

$$V_i = \frac{1}{2}\|l_i - l_{d_i}\|^2 + \|A_i - A_{d_i}\|^2.$$

(59)

It is easy to see that the control for the ith satellite with $i \leq m - 1$ is the same as the control for the (i, m) pair of satellites. The initial value of V should be less than the minimum value of c_i for $i = 1, 2, \ldots, m - 1$. Then the solution will converge to the set where

$$l_i = l_{d_i}$$
$$A_i = A_{d_i}$$
$$M_i - M_n = \phi_i$$

(60)

are satisfied for $i = 1, 2, \ldots, m$.

TRANSFER OF SATELLITE FORMATIONS TO CIRCULAR ORBITS

Formations on circular orbits are of great importance.[6] If the destination of at least one of the satellites in a formation is circular, there is difficulty in determining the mean anomaly. Since the Laplace vector A vanishes, the perigee point cannot be determined. In this case, the position of a satellite on the circular orbit is usually described by using the argument of latitude, defined as the angle θ between the line

of ascending node and the position vector q. It is easy to see that on an elliptic orbit, this angle is

$$\theta = \omega + f, \tag{61}$$

where ω is the argument of perigee and f is the true anomaly. On a circular orbit, we have

$$\theta = n(t - \tau), \tag{62}$$

where τ is the time when the satellite passes the ascending node for the first time. As can be seen, θ serves as the mean anomaly for circular orbit.

Although θ and $\dot{\theta}$ can be easily determined, as discussed in the APPENDIX, they are not suitable for our Lyapunov function based control laws because θ is not affine in time on an elliptic orbit. Thus, we use the angle M_θ, which is the mean anomaly measured from the line of node. M_θ can be determined from θ, but if the orbit is close to circular, we can use θ as an approximation for M_θ. Our analysis in the APPENDIX shows that the derivative of M_θ is

$$
\begin{aligned}
\dot{M}_\theta = {} & n - \frac{r_\omega^2}{a^2\sqrt{1-e^2}} \frac{\cos(I)}{h\sin(I)} \left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u \\
& - \left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1-e^2)}\right) \frac{1}{\mu}\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u \\
& + \frac{r^2 - r_\omega^2}{\mu a^2 e\sqrt{1-e^2}} \left(\left(\frac{\partial A}{\partial p}\right)^T \hat{A}_\perp \cdot u\right).
\end{aligned}
\tag{63}
$$

Here, ω is the argument of perigee, E and E_ω are the eccentric anomalies corresponding to $\theta - \omega$ and ω, r_ω denotes the distance between the ascending node and the gravitational center. We let $\hat{n} = (l_x^2 + l_y^2)^{-1/2}[-l_y, l_x, 0]^T$, where $\hat{n}$ is the unit vector pointing to the ascending node. Let $\hat{A}_\perp$ be the unit vector perpendicular to $\hat{A}$ and l forming an acute angle with $\hat{n}$.

We now study the case when we want to set up a formation in which all the members are on circular orbits with the same radius. According to the previous discussion under PERIODIC FORMATION, this formation is a periodic formation. Let τ_i be the time when the ith satellite passes the line of ascending node for the first time. Similar to the perigee passage time in our elliptic orbit case, τ_i is a constant of the free motion for satellite i. Hence, by specifying the differences $\tau_i - \tau_j$ or $\theta_i - \theta_j$ between satellites, a periodic formation on circular orbits can be uniquely determined.

Similar to the elliptic orbit case, we introduce a variable ψ_i, which is defined as

$$
\psi_i = \begin{cases}
\left(\dfrac{a_i}{a_d}\right)^{3/2} M_{\theta_i}, & \text{if } \theta_i \in (-\varepsilon, \pi + \varepsilon) \\[3mm]
\left(\dfrac{a_i}{a_d}\right)^{3/2} (2\pi - M_{\theta_i}), & \text{if } \theta_i \in (\pi - \varepsilon, 2\pi + \varepsilon)
\end{cases}
\tag{64}
$$

for $i = 1, 2$, where a_d is the common length of the semimajor axes (or radius) for the destination orbits. Here, the reasons for using two charts are the same as in the case of elliptic orbits.

Our calculations in the APPENDIX show that the derivative of ψ is

$$\dot{\psi} \;=\; \sqrt{\frac{\mu}{a_d^3}} + \left(\frac{\partial l}{\partial p}\right)^T (\lambda \hat{n} + \kappa l) \cdot u + \left(\frac{\partial A}{\partial p}\right)^T (\sigma \hat{A} + \varrho \hat{A}_\perp) \cdot u \tag{65}$$

where for $\theta \in (-\varepsilon, \pi + \varepsilon)$,

$$\lambda \;=\; -\left(\frac{a}{a_d}\right)^{3/2} \frac{r_\omega^2}{a^2\sqrt{1-e^2}} \frac{\cos(I)}{h\sin(I)} \approx -\left(\frac{a}{a_d}\right)^{3/2} \frac{r_\omega^2}{a} \frac{\cos(I)}{h\sin^2(I)}$$

$$\kappa \;=\; \frac{3}{\mu a(1-e^2)}\psi$$

$$\sigma \;=\; \frac{3e}{\mu(1-e^2)}\psi - \frac{1}{\mu}\left(\frac{a}{a_d}\right)^{3/2}\left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1-e^2)}\right) \tag{66}$$

$$\approx -\frac{1}{\mu}\left(\frac{a}{a_d}\right)^{3/2}\left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1-e^2)}\right)$$

$$\varrho \;=\; \left(\frac{a}{a_d}\right)^{3/2} \frac{r^2 - r_\omega^2}{\mu a^2 e\sqrt{1-e^2}} \approx -\left(\frac{a}{a_d}\right)^{3/2} \frac{2(\cos(E) - \cos(E_\omega))}{\mu},$$

and for $\theta \in (\pi - \varepsilon, 2\pi + \varepsilon)$,

$$\lambda \;=\; \left(\frac{a}{a_d}\right)^{3/2} \frac{r_\omega^2}{a^2\sqrt{1-e^2}} \frac{\cos(I)}{h\sin(I)} \approx \left(\frac{a}{a_d}\right)^{3/2} \frac{r_\omega^2}{a^2} \frac{\cos(I)}{h\sin(I)}$$

$$\kappa \;=\; \frac{3}{\mu a(1-e^2)}\psi$$

$$\sigma \;=\; \frac{3e}{\mu(1-e^2)}\psi + \frac{1}{\mu}\left(\frac{a}{a_d}\right)^{3/2}\left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1-e^2)}\right) \tag{67}$$

$$\approx \frac{1}{\mu}\left(\frac{a}{a_d}\right)^{3/2}\left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1-e^2)}\right)$$

$$\varrho \;=\; -\left(\frac{a}{a_d}\right)^{3/2} \frac{r^2 - r_\omega^2}{\mu a^2 e\sqrt{1-e^2}} \approx \left(\frac{a}{a_d}\right)^{3/2} \frac{2(\cos(E) - \cos(E_\omega))}{\mu}.$$

We use the symbol $\approx$ when the eccentricity of the orbits are very small. When the orbit is circular, E and E_ω are not well defined, but we have $M_\theta = \theta$ and

$$\dot{\theta} \;=\; n - \frac{\cos(I)}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u. \tag{68}$$

Thus, when $e = 0$

$$\lambda \;=\; \mp\left(\frac{a}{a_d}\right)^{3/2} \frac{\cos(I)}{h\sin(I)}$$

$$\kappa \;=\; \frac{3}{\mu a(1-e^2)}\psi \tag{69}$$

$$\sigma \;=\; 0$$

$$\varrho \;=\; 0.$$

This result agrees with the limits of **(66)** and **(67)** as $e \to 0$.

Mimicking what was done in the elliptic orbit case, we design a Lyapunov function on the phase space of two satellites as follows:

$$V = \begin{cases} V_1 + V_2 + 4\sin\left(\dfrac{\Psi_1 - \Psi_2 - \phi}{4}\right)^2, & \text{if } \theta_i \in (-\varepsilon, \pi + \varepsilon) \\[2ex] V_1 + V_2 + 4\sin\left(\dfrac{\Psi_1 - \Psi_2 + \phi}{4}\right)^2, & \text{if } \theta_i \in (\pi - \varepsilon, 2\pi + \varepsilon) \end{cases} \tag{70}$$

where

$$V_1 = \frac{1}{2}(\|l_1 - l_{d_1}\|^2 + \|A_1\|^2)$$
$$V_2 = \frac{1}{2}(\|l_2 - l_{d_2}\|^2 + \|A_2\|^2). \tag{71}$$

The derivative of this function has similar form to Equation **(28)**. The control law u such that $\dot{V} \leq 0$ can be chosen as

$$\begin{aligned} u_1 &= -\left[\left(l_1 - l_{d_1} + \sin\left(\frac{\Psi_1 - \Psi_2 \mp \phi}{2}\right)(\lambda_1 \hat{n}_1 + \kappa_1 l_1)\right) \times q_1 \right. \\ &\quad + l_1 \times \left(A_1 + \sin\left(\frac{\Psi_1 - \Psi_2 \mp \phi}{2}\right)(\sigma_1 \hat{A}_1 + \varrho_1 \hat{A}_{1\perp})\right) \\ &\quad \left. + \left(\left(A_1 + \sin\left(\frac{\Psi_1 - \Psi_2 \mp \phi}{2}\right)(\sigma_1 \hat{A}_1 + \varrho_1 \hat{A}_{1\perp})\right) \times p_1\right) \times q_1 \right] \\ u_2 &= -\left[\left(l_2 - l_{d_2} + \sin\left(\frac{\Psi_1 - \Psi_2 \mp \phi}{2}\right)(\lambda_2 \hat{n}_2 + \kappa_2 l_2)\right) \times q_2 \right. \\ &\quad + l_2 \times \left(A_2 - \sin\left(\frac{\Psi_1 - \Psi_2 \mp \phi}{2}\right)(\sigma_2 \hat{A}_2 + \varrho_2 \hat{A}_{2\perp})\right) \\ &\quad \left. + \left(\left(A_2 - \sin\left(\frac{\Psi_1 - \Psi_2 \mp \phi}{2}\right)(\sigma_2 \hat{A}_2 + \varrho_2 \hat{A}_{2\perp})\right) \times p_2\right) \times q_2 \right]. \end{aligned} \tag{72}$$

Based on Lemma 4, the following lemma is obvious.

Lemma 6:
Let
$$c < \min\{c_1, c_2\} \tag{73}$$

where

$$c_i = \min\left\{\frac{1}{2}\|l_{d_i}\|^2, \frac{1}{2}\mu^2\right\} \tag{74}$$

for $i = 1, 2$. Then the set

$$S_M = \{z \mid V(z) \leq c\} \tag{75}$$

is a compact subset of $\Sigma_{e_1} \times \Sigma_{e_2}$.

Hence we prove that, starting within the set S_M, the controlled dynamics will converge to the maximal invariant set where formation relationships are obtained.

Proposition 7:

With V, c, and u_i given as in **(70)**, **(73)**, and **(72)**, the trajectory starting from the point $(q_{10}, p_{10}, q_{20}, p_{20})$ that satisfies

$$V(q_{10}, p_{10}, q_{20}, p_{20}) \leq c \tag{76}$$

will converge to the set for which

$$l_i = l_{d_i}$$
$$A_i = 0 \tag{77}$$
$$\theta_1 - \theta_2 = \phi$$

are satisfied for $i = 1, 2$.

Proof:

Let $u_1 = 0$, and take the inner product with q_1 on both sides of the equation to obtain

$$(l_1 \times B_1) \cdot q_1 = 0, \tag{78}$$

where

$$B_1 = A_1 + \sin\left(\frac{\psi_1 - \psi_2 \mp \phi}{2}\right)(\sigma_1 \hat{A}_1 + \varrho_1 \hat{A}_{1\perp}). \tag{79}$$

At the time t_0 when $E_1(t_0) = E_{\omega_1}$, since $M_{\theta_1} = 0$ and $r_1 = r_{\omega_1}$, we have

$$\varrho_1(t_0) = 0 \text{ and } \sigma_1(t_0) = 0, \tag{80}$$

hence

$$B_1(t_0) = A_1. \tag{81}$$

On the other hand,

$$(q_1 \times l_1) \cdot B_1 = 0 \tag{82}$$

tells us that B_1 lies in the plane spanned by l_1 and q_1. Thus, when $q_1(t_0)$ is not aligned with A_1, $B_1(t_0)$ must vanish. This implies that

$$A_1 = 0. \tag{83}$$

However, it is also possible that $q_1(t_0)$ is aligned with A_1. In this case we have

$$E_{\omega_1} = 0 \text{ and } \omega_1 = 0. \tag{84}$$

Therefore, for all time t,

$$\varrho_1(t) = \pm\left(\frac{a_1}{a_d}\right)^{3/2} \frac{r_1^2 - r_{\omega_1}^2}{\mu a_1^2 e_1 \sqrt{1 - e_1^2}}$$

$$\sigma_1(t) = \frac{3e_1}{\mu(1 - e_1^2)} \psi_1 - \frac{1}{\mu}\left(\frac{a_1}{a_d}\right)^{3/2} \sin(E_1)\left(1 + \frac{r_1}{a_1(1 - e_1^2)}\right). \tag{85}$$

Thus,

$$B_1(t) = A_1 + \sin\left(\frac{\psi_1 - \psi_2 \mp \phi}{2}\right)(\sigma_1 \hat{A}_1 + \varrho_1 \hat{A}_{1\perp}), \tag{86}$$

and $B_1(t)$ has to be aligned with $q_1(t)$ for all time t. If we let T_1 be the orbital period of the first satellite and T_2 be the orbital period of the second satellite, then after one period T_1 the first satellite will return to the same position, which implies that

$$A_1 + \sin\left(\frac{\psi_1(t + T_1) - \psi_2(t + T_1) \mp \phi}{2}\right)(\sigma_1(t + T_1)\hat{A}_1 + \varrho_1(t + T_1)\hat{A}_{1\perp})$$

$$= A_1 + \sin\left(\frac{\psi_1(t) - \psi_2(t) \mp \phi}{2}\right)(\sigma_1(t)\hat{A}_1 + \varrho_1(t)\hat{A}_{1\perp}). \tag{87}$$

Because $M_{\theta_1}(t) = M_{\theta_1}(t + T_1)$ and $E_1(t) = E_1(t + T_1)$, we know $\sigma_1(t + T_1) = \sigma_1(t)$ and $\varrho_1(t + T_1) = \varrho_1(t)$. Thus, the above equation results in

$$\sin\left(\frac{\left(\frac{a_1}{a_d}\right)^{3/2} M_{\theta_1}(t) - \left(\frac{a_2}{a_d}\right)^{3/2} M_{\theta_2}(t) - \phi}{2}\right)$$

$$= \sin\left(\frac{\left(\frac{a_1}{a_d}\right)^{3/2} M_{\theta_1}(t) - \left(\frac{a_2}{a_d}\right)^{3/2} M_{\theta_2}(t + T_1) - \phi}{2}\right). \tag{88}$$

Therefore, we must have

$$M_{\theta_2}(t) = M_{\theta_2}(t + T_1). \tag{89}$$

Hence, T_1/T_2 is a positive integer. On the other hand, the above arguments can be applied to the second satellite. If $A_2 \neq 0$, T_2/T_1 must be a positive integer. Hence, $T_2 = T_1$, which implies that $a_1 = a_2$.

Still assuming $a_1(t_0)$ is aligned with A_1, we know that by a property of elliptic orbits, after half period, $q_1(t_0 + T_1/2)$ is aligned with A_1 again, which means that

$$\sin\left(\frac{\psi_1(t_0 + T_1/2) - \psi_2(t_0 + T_1/2) \mp \phi}{2}\right)\varrho_1(t_0 + T_1/2) = 0. \tag{90}$$

According to the definition of ψ, one can verify that

$$\sin\left(\frac{\psi_1(t + T_1/2) - \psi_2(t + T_1/2) \mp \phi}{2}\right) = -\sin\left(\frac{\psi_1(t) - \psi_2(t) \mp \phi}{2}\right). \tag{91}$$

Therefore, the first possibility is that

$$\varsigma_1(t_0 + T_1/2) = -\varrho_1(t_0) = 0. \tag{92}$$

This further implies that $r_1(t_0 + T_1/2) = r_1(t_0)$. Because at time t_0 the satellite is at perigee and at time $t_0 + T_1/2$ the satellite is at apogee, we conclude that the orbit is circular; that is, $A_1 = 0$.

The second possibility for Equation **(90)** to hold is that

$$\sin\left(\frac{\psi_1(t_0) - \psi_2(t_0) \mp \phi}{2}\right) = 0. \tag{93}$$

However, because $T_1 = T_2$, we must have

$$\psi_1(t) - \psi_2(t) = \psi_1(t_0) - \psi_2(t_0), \tag{94}$$

which implies that

$$\sin\left(\frac{\psi_1(t) - \psi_2(t) \mp \phi}{2}\right) = 0. \tag{95}$$

One can make the same argument as in the proof of the single satellite case[3] to show that

$$l_1 = l_{d_1}$$
$$A_1 = 0. \tag{96}$$

To this point, we have proved that our algorithm will cause either A_1 or A_2 to vanish. Without loss of generality, we proceed by assuming $A_1 = 0$. Then, from $u_1 = 0$, we get

$$\left(l_1 - l_{d_1} + \sin\left(\frac{\psi_1 - \psi_2 \mp \phi}{2}\right)(\lambda_1 \hat{n}_1 + \kappa_1 l_1)\right) \times q_1 = 0. \tag{97}$$

Let

$$D_1 = l_1 - l_{d_1} + \sin\left(\frac{\psi_1 - \psi_2 \mp \phi}{2}\right)(\lambda_1 \hat{n}_1 + \kappa_1 l_1). \tag{98}$$

D_1 must be aligned with q_1 and also lies in the plane passing $(l_1 - l_{d_1})$ spanned by l_1 and n_1. Because the plane is fixed and q_1 is periodic in time, at time t and time $(t + T_1)$, the intersection of q_1 and the plane must be the same. This tells us that T_1/T_2 is a positive integer. Now, if $A_2 = 0$, the above arguments can be repeated so that we conclude T_2/T_1 is also a positive integer. If $A_2 \neq 0$, we can repeat the arguments following Equation (84) to conclude that T_2/T_1 is also a positive integer. Therefore, we conclude that $T_1 = T_2$ and $a_1 = a_2$.

According to a property of circular orbits, at times t and $(t + T_1/2)$, the intersection of q_1 and the plane spanned by l_1 and n_1 passing $l_1 - ld_1$ should be the same, that is,

$$D_1(t) = D_1(t + T_1/2). \tag{99}$$

Thus,

$$\sin\left(\frac{\psi_1(t) - \psi_2(t) \mp \phi}{2}\right)(\lambda_1(t)\hat{n}_1 + \kappa_1(t)l_1)$$
$$= \sin\left(\frac{\psi_1(t + T_1/2) - \psi_2(t + T_1/2) \mp \phi}{2}\right)(\lambda_1(t + T_1/2)\hat{n}_1 + \kappa_1(t + T_1/2)l_1). \tag{100}$$

Because of (91), Equation (100) is satisfied only if

$$\sin\left(\frac{\psi_1(t) - \psi_2(t) \mp \phi}{2}\right) = 0 \tag{101}$$

or

$$\lambda_1(t)\hat{n}_1 + \kappa_1(t)l_1 = -\lambda_1(t + T_1/2)\hat{n}_1 - \kappa_1(t + T_1/2)l_1. \tag{102}$$

Equation (102) implies that for all time t,

$$\kappa(t) = -\kappa(t + T_1/2), \tag{103}$$

which further implies that

$$M_{\theta_1}(t) = \frac{\pi}{2}. \tag{104}$$

This is impossible. Thus, the only possibility is that

$$\sin\left(\frac{\psi_1(t) - \psi_2(t) \mp \phi}{2}\right) = 0 \tag{105}$$

Because of **(105)**, the time varying parts in the equation $u_1 = 0$ vanish. We can make the same argument as in the proof for the single satellite case[3] to show that

$$l_1 = l_{d_1}$$
$$A_1 = 0. \tag{106}$$

Now let $u_2(t) = 0$ and substitute Equation **(105)**. We conclude that the second satellite will achieve

$$l_2 = l_{d_2}$$
$$A_2 = 0. \tag{107}$$

Thus, we have

$$l_i = l_{d_i}$$
$$A_i = 0 \tag{108}$$
$$\psi_1 - \psi_2 = \pm\phi$$

for $i = 1,2$. By definition of ψ_1 and ψ_2 in **(64)**, we have

$$\left(\frac{a_1}{a_d}\right)^{3/2} M_{\theta_1} - \left(\frac{a_2}{a_d}\right)^{3/2} M_{\theta_2} = \phi. \tag{109}$$

However, we already know $a_1 = a_2 = a_d$ and that the orbits are circular; thus, we conclude that

$$\theta_1 - \theta_2 = \phi. \tag{110}$$

∎

We mention that this algorithm is not limited to set up formations on circular orbits. With slight modification to the proof one can show the convergence result holds for elliptic orbits.

SIMULATION RESULTS

To verify our algorithms, a series of simulations were carried out using Matlab. First, we show a controlled transfer of two satellites from orbit $[a,e,i,\omega,\Omega] = [20,0.1,\pi/4,\pi/2,0]$ with initial separation of mean anomaly $\pi/90$ to the orbit $[a,e,i,\omega,\Omega] = [20,0.05,\pi/3,\pi/2,0]$ with final separation of mean anomaly $\pi/18 \approx 0.175$. Only relative motion between the satellites is plotted in an inertial frame centered at one of the satellites. FIGURE 2 displays the desired relative motion between the satellites. FIGURE 3 displays the relative motion between the satellites using the proposed control algorithm. The final orbits for the two satellites are:

$$[a_1, e_1, I_1, \omega_1, \Omega_1] = [25.10, 0.0495, 1.0472, 1.6231, 0]$$
$$[a_2, e_2, I_2, \omega_2, \Omega_2] = [25.12, 0.0507, 1.0472, 1.6231, 0]$$

and the final separation of mean anomaly is 0.168. As can be seen, the desired orbit and separation are achieved with small error. We noticed that by adjusting the
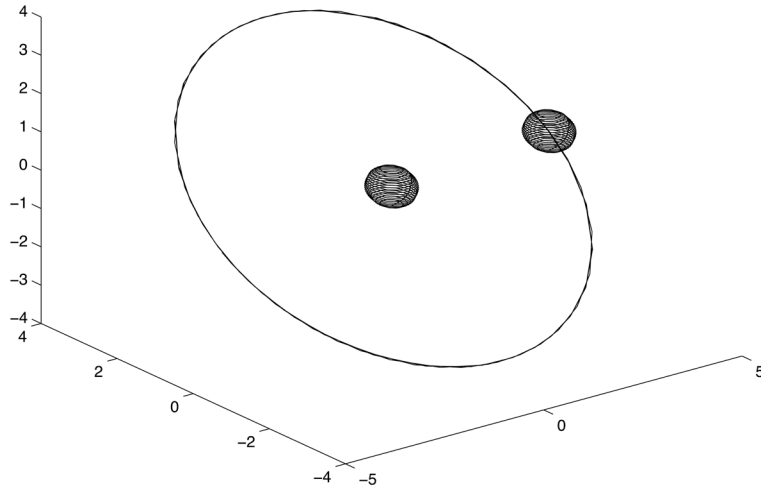
FIGURE 2. The desired final relative motion of two satellites (length unit, one tenth of Earth radius).
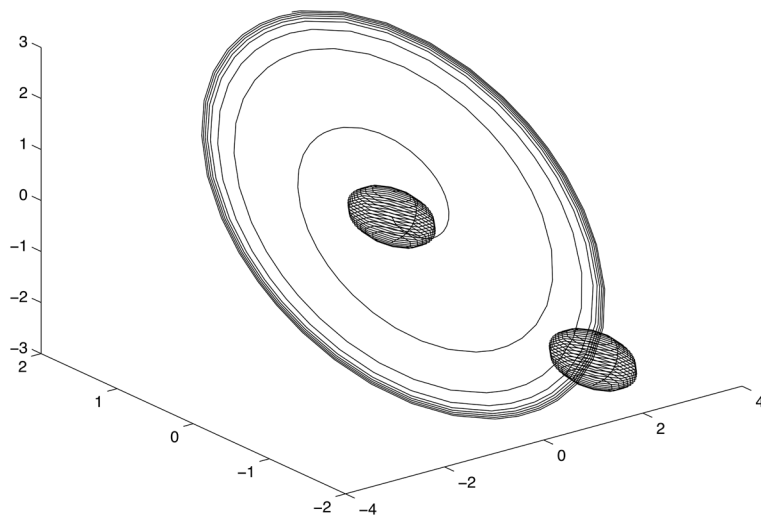

FIGURE 3. The relative motion achieved by our control law for elliptic orbits (length unit, one tenth of Earth radius).

weight of the terms in our Lyapunov functions, we can control the distribution of errors among the orbital elements of final orbits.

Similar experiments were performed to set up a two satellite formation in circular orbits. Initially, the two satellites were on an elliptic orbit with $[a,e,i,\omega,\Omega] = [20,0.1,\pi/4,\pi/2,0]$ and the separation of argument of latitude equal to $\pi/90$. Their destination orbit was circular with $[a,e,i,\omega,\Omega] = [25,0,\pi/3,\pi/4]$ and separation $\pi/18$. Using our algorithm, the final orbits obtained are:

$$[a_1,e_1,I_1,\Omega_1] = [25.082, 0.0023, 1.0472, 0.7854]$$

$$[a_2,e_2,I_2,\Omega_2] = [25.078, 0.0022, 1.0472, 0.7854]$$

and the final separation equal to 0.174. Again, the formation is achieved with small error.

SUMMARY AND FUTURE DIRECTIONS

In this paper we have proposed control algorithms that can be used to set up periodic formations of satellites on elliptic and circular orbits. The shape space formed by the angular momentum vectors and Laplace vectors is appropriate to describe satellite formations. The control laws we propose are based on a Lyapunov function on this shape space and proved to be convergent. We have not considered the effect of perturbations, such as the J_2 effect. This is currently being investigated.

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APPENDIX

In this appendix we derive the perturbation equations for the orbital elements used in our control algorithm. The control forces are continuous functions of time. Hence, the derivatives are taken along the *controlled* dynamics of a satellite.

Starting with the eccentricity, we have

$$\dot{e}(t) = \frac{A \cdot \dot{A}}{\mu \|A\|} = \frac{A}{\mu \|A\|} \cdot \frac{\partial A}{\partial p} u(t) = \frac{1}{\mu}\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u(t), \tag{111}$$

where $\hat{A}$ is the unit vector along the direction of A. We also have

$$\dot{h}(t) = \frac{l \cdot l}{\|l\|} = \frac{1}{h}\left(\frac{\partial l}{\partial p}\right)^T l \cdot u(t) \tag{112}$$

and

$$\dot{a}(t) = \frac{1}{\mu(1 - e^2)}(2h\dot{h} + 2\mu ae\dot{e})$$

$$= \frac{2}{\mu(1 - e^2)}\left(\frac{\partial l}{\partial p}\right)^T l \cdot u(t) + \frac{2ae}{\mu(1 - e^2)}\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u(t). \tag{113}$$

We now derive the formulæ for $\dot{M}$. Since $r^2 = q \cdot q$, we have $r\dot{r} = q \cdot p$. Hence

$$h^2 = l \cdot l = (q \times p) \cdot (q \times p)$$

$$= (q \cdot q)(p \cdot p) - (q \cdot p)(q \cdot p) = r^2\mu\left(\frac{2}{r} - \frac{1}{a}\right) - r^2\dot{r}^2. \tag{114}$$

On the other hand

$$h^2 = \mu a(1 - e^2). \tag{115}$$

Thus,

$$\dot{r}^2 = -\frac{\mu a(1 - e^2)}{r} + \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{116}$$

After applying $r = a(1 - e\cos(E))$, on simplification we obtain

$$\dot{r} = \sqrt{\mu a}\frac{e\sin(E)}{r}. \tag{117}$$

Taking derivatives on both sides of $r = a(1 - e\cos(E))$,

$$\dot{r} = \dot{a}(1 - e\cos(E)) + a(\dot{e}\cos(E) - e\sin(E)\dot{E}). \tag{118}$$

Thus,

$$\dot{E} = \frac{\dot{r}}{ae\sin(E)} - \frac{r}{a^2e\sin(E)}\dot{a} + \frac{a\cos(E)}{ae\sin(E)}\dot{e}$$

$$= \sqrt{\frac{\mu}{a}}\frac{1}{r} + \frac{\cos(E)}{e\sin(E)}\dot{e} - \frac{r}{a^2e\sin(E)}\dot{a}. \tag{119}$$

Taking derivatives on both sides of $M = E - e\sin(E)$,

$$\dot{M} = \dot{E} - \dot{e}\sin(E) - e\cos(E)\dot{E} = (1 - e\cos(E))\dot{E} - \dot{e}\sin(E). \tag{120}$$

Combining this equation with Equations **(119)**, **(113)**, and **(111)**, we obtain

$$\dot{M} = (1 - e\cos(E))\sqrt{\frac{\mu}{a}}\frac{1}{r} + \left(\frac{\cos(E)(1 - e\cos(E))}{e\sin(E)} - \sin(E)\right)\dot{e} - \frac{r(1 - e\cos(E))}{a^2 e\sin(E)}\dot{a}$$

$$= \sqrt{\frac{\mu}{a^3}} + \frac{\cos(E) - e}{e\sin(E)}\dot{e} - \frac{(1 - e\cos(E))^2}{ae\sin(E)}\dot{a} \tag{121}$$

$$= n + \eta\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u(t) + \xi\left(\frac{\partial l}{\partial o}\right)^T l \cdot u(t),$$

where $n = \sqrt{\mu/a^3}$ and

$$\xi(l, A, E) = -\frac{2(1 - e\cos(E))^2}{\mu ae(1 - e^2)\sin(E)}$$

$$\eta(l, A, E) = \frac{\cos(E) - e}{\mu e\sin(E)} - \frac{2(1 - e\cos(E))^2}{\mu(1 - e^2)\sin(E)}. \tag{122}$$

The inclination I is defined as the angle between the z-axis and the angular momentum vector l. It can be determined from

$$\cos(I) = \frac{l_z}{h}. \tag{123}$$

Taking derivatives on both sides, we obtain

$$-\sin(I)\dot{I} = \frac{l_z l_x \dot{l}_x + l_z l_y \dot{l}_y - (l_x^2 + l_y^2)\dot{l}_z}{h^3}$$

$$= -\frac{\sin(I)}{h}[\cos(I)\sin(\Omega), -\cos(I)\cos(\Omega), -\sin(I)]\dot{l}, \tag{124}$$

because

$$\cos(\Omega) = \frac{-l_y}{\sqrt{l_x^2 + l_y^2}} = \frac{-l_y}{h\sin(I)}. \tag{125}$$

Thus,

$$\dot{I} = \frac{1}{h}[\cos(I)\sin(\Omega), -\cos(I)\cos(\Omega), -\sin(I)]\left(\frac{\partial l}{\partial p}\right)u. \tag{126}$$

The argument of the ascending node Ω is defined as the angle between the x-axis and the ascending node. It can be determined from

$$\tan(\Omega) = -\frac{l_x}{l_y}. \tag{127}$$

By taking derivatives on both sides of **(127)**, we obtain

$$\frac{1}{\cos^2(\Omega)}\dot{\Omega} = -\frac{l_y \dot{l}_x - l_x \dot{l}_y}{l_y^2} = -\frac{1}{l_y^2}[l_y, -l_x, 0]\frac{\partial l}{\partial p}u. \tag{128}$$

Let $\hat{n} = (l_x^2 - l_y^2)^{-1/2}[-l_y, l_x, 0]^T$, then

$$\dot{\Omega} = \frac{h\sin(I)\cos^2(\Omega)}{l_y^2}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u = \frac{1}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u. \tag{129}$$

Note that $\hat{n}$ is the unit vector pointing to the ascending node.

The argument of the perigee ω is defined as the angle between the vector $\hat{n}$ and $\hat{A}$. It can be determined from

$$\cos(\omega) = \hat{n} \cdot \hat{A}. \tag{130}$$

Taking derivatives on both sides,

$$\dot{\omega} = -\frac{1}{\sin(\omega)}(\dot{\hat{n}} \cdot \hat{A} + \dot{\hat{A}} \cdot \hat{n}). \tag{131}$$

Because

$$\dot{\hat{A}} = \frac{1}{\mu e}(Id - \hat{A}\hat{A}^T)\dot{A} \tag{132}$$

and

$$\dot{\hat{n}} = \frac{1}{h\sin(I)}(Id - \hat{n}\hat{n}^T)\dot{n}, \tag{133}$$

where Id is the 3×3 identity matrix, one can verify that

$$\dot{\hat{n}} \cdot \hat{A} = \dot{\Omega}\cos(I)\sin(\omega) \tag{134}$$

and

$$\dot{\hat{A}} \cdot \hat{n} = \frac{1}{\mu e}(\hat{n} - \cos(\omega)\hat{A}) \cdot \dot{A} = \frac{1}{\mu e}\sin(\omega)\hat{A}_\perp \cdot \dot{A} \tag{135}$$

where $\hat{A}_\perp$ is the unit vector perpendicular to $\hat{A}$ and l forming an acute angle with $\hat{n}$. The derivative of ω is

$$\dot{\omega} = -\frac{\cos(I)}{h^2\sin^2(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u - \frac{1}{\mu e}\left(\frac{\partial A}{\partial p}\right)^T \hat{A}_\perp \cdot u. \tag{136}$$

The longitude parameter θ is used instead of the true anomaly when the orbit is circular. To determine θ, we have

$$q \cdot \hat{n} = q_x\cos(\Omega) + q_y\sin(\Omega) = r\cos(\theta). \tag{137}$$

Hence,

$$\cos(\theta) = \frac{x}{r}\cos(\Omega) + \frac{y}{r}\sin(\Omega). \tag{138}$$

On the other hand,

$$-q \times \hat{n} = r\sin(\theta)\frac{l}{h}. \tag{139}$$

This gives the following set of equations:

$$\sin(\theta)\frac{l_x}{h} = \frac{z}{r}\sin(\Omega)$$

$$\sin(\theta)\frac{l_y}{h} = -\frac{z}{r}\cos(\Omega) \tag{140}$$

$$\sin(\theta)\frac{l_z}{h} = -\frac{x}{r}\sin(\Omega) + \frac{y}{r}\cos(\Omega).$$

Thus, θ can be uniquely determined by using Equation **(138)** and any one of **(140)**.

We now calculate the derivative of θ, which contains two parts. The first part is due to the orbital motion of the satellite and the second part is due to the perturbation of the line of ascending node; that is,

$$\dot{\theta} = \frac{h}{r^2} - \frac{\cos(I)}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u. \qquad (141)$$

We now derive Equation **(63)** for $\dot{M}_\theta$. We start from the cosine relationship between the true anomaly f and the eccentric anomaly E

$$\cos(E) = \frac{e + \cos(f)}{1 + e\cos(f)}. \qquad (142)$$

Taking derivatives on both sides yields

$$-\sin(E)\dot{E} = \frac{(e^2 - 1)\sin(f)}{(1 + e\cos(f))^2}\dot{f} + \frac{\sin^2(f)}{(1 + e\cos(f))^2}\dot{e}. \qquad (143)$$

Because

$$\sin(E) = \frac{\sqrt{1 - e^2}\sin(f)}{1 + e\cos(f)}, \qquad (144)$$

we have

$$\dot{E} = \frac{\sqrt{1 - e^2}}{1 + e\cos(f)}\dot{f} - \frac{\sin(E)}{1 - e^2}\dot{e} = \frac{1 - e\cos(E)}{\sqrt{1 - e^2}}\dot{f} - \frac{\sin(E)}{1 - e^2}\dot{e}. \qquad (145)$$

We know that

$$f = \theta - \omega. \qquad (146)$$

The true anomaly of the ascending node is $-\omega$. Let E_ω be the eccentric anomaly of the ascending node. Its derivative is

$$\dot{E}_\omega = \frac{1 - e\cos(E_\omega)}{\sqrt{1 - e^2}}(-\dot{\omega}) - \frac{\sin(E_\omega)}{1 - e^2}\dot{e}. \qquad (147)$$

Let M_θ be the mean anomaly measured from the line of ascending node. By using Kepler's equation, it is easy to see that

$$M_\theta = E - e\sin(E) - (E_\omega - e\sin(E_\omega)). \qquad (148)$$

Thus, the derivative is

$$\dot{M}_\theta = (1 - e\cos(E))\dot{E} - (1 - e\cos(E_\omega))\dot{E}_\omega - \dot{e}(\sin(E) - \sin(E_\omega))$$

$$= \frac{(1 - e\cos(E))^2}{\sqrt{1 - e^2}}\dot{\theta} - \dot{e}(\sin(E) - \sin(E_\omega))$$

$$- \dot{e}\left(\frac{(1 - e\cos(E))\sin(E)}{1 - e^2} - \frac{(1 - e\cos(E_\omega))\sin(E_\omega)}{1 - e^2}\right) \qquad (149)$$

$$- \frac{(1 - e\cos(E))^2 - (1 - e\cos(E_\omega))^2}{\sqrt{1 - e^2}}\dot{\omega}.$$

We know that

$$(1 - e\cos(E)) = \frac{r}{a}. \qquad (150)$$

Therefore,

$$\frac{(1 - e\cos(E))^2 - (1 - e\cos(E_\omega))^2}{\sqrt{1 - e^2}} = \frac{r^2 - r_\omega^2}{a^2\sqrt{1 - e^2}}. \tag{151}$$

Because of Equation **(136)**,

$$\begin{aligned}
\frac{r^2 - r_\omega^2}{a^2\sqrt{1 - e^2}}\dot{\omega} &= -\frac{r^2 - r_\omega^2}{\mu a^2 e\sqrt{1 - e^2}}\left(\left(\frac{\partial A}{\partial p}\right)^T \hat{A}_\perp \cdot u\right) \\
&\quad - \frac{r^2 - r_\omega^2}{a^2\sqrt{1 - e^2}}\frac{\cos(I)}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u.
\end{aligned} \tag{152}$$

Since we already know that

$$\dot{\theta} = \frac{h}{r^2} - \dot{\Omega}\cos(I)$$

$$\dot{e} = \frac{1}{\mu}\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u(t), \tag{153}$$

substitute the derivatives in **(149)** from **(152)** and **(153)**, to obtain

$$\begin{aligned}
\dot{M}_\theta &= n - \frac{r^2}{a^2\sqrt{1 - e^2}}\frac{\cos(I)}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u \\
&\quad - \left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1 - e^2)}\right)\frac{1}{\mu}\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u \\
&\quad - \frac{r^2 - r_\omega^2}{\mu a^2 e\sqrt{1 - e^2}}\left(\left(\frac{\partial A}{\partial p}\right)^T \hat{A}_\perp \cdot u\right) \\
&\quad + \frac{r^2 - r_\omega^2}{a^2\sqrt{1 - e^2}}\frac{\cos(I)}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u,
\end{aligned} \tag{154}$$

which simplifies to

$$\begin{aligned}
\dot{M}_\theta &= n - \frac{r_\omega^2}{a^2\sqrt{1 - e^2}}\frac{\cos(I)}{h\sin(I)}\left(\frac{\partial l}{\partial p}\right)^T \hat{n} \cdot u \\
&\quad - \left(\sin(E) - \sin(E_\omega) + \frac{r\sin(E) - r_\omega\sin(E_\omega)}{a(1 - e^2)}\right)\frac{1}{\mu}\left(\frac{\partial A}{\partial p}\right)^T \hat{A} \cdot u \\
&\quad + \frac{r^2 - r_\omega^2}{\mu a^2 e\sqrt{1 - e^2}}\left(\left(\frac{\partial A}{\partial p}\right)^T \hat{A}_\perp \cdot u\right).
\end{aligned} \tag{155}$$