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LIE–POISSON STRUCTURES, DUAL-SPIN SPACECRAFT AND ASYMPTOTIC STABILITY*

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1. INTRODUCTION

A DUAL-SPIN spacecraft may be viewed as a simple *spinning* platform carrying a motor-driven symmetric rigid rotor (see Fig. 1). The rotor is spun up to a desired angular velocity relative to the platform and then it is maintained at this constant angular velocity. As we shall show below, this requires that the motor continue to exert a nonzero torque (dependent on the

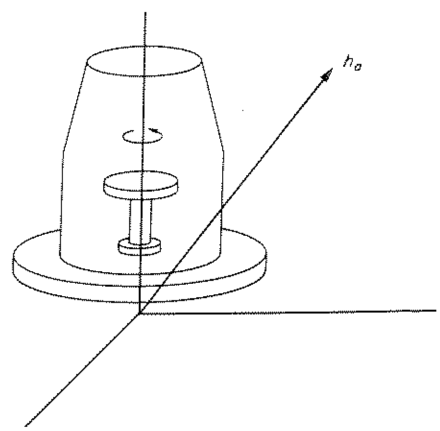

FIG. 1.

platform angular velocity). The essential intuition here is that in the presence of a suitable damping mechanism, and for sufficiently high rotor velocities, one can expect the spacecraft angular momentum vector to align itself eventually with the rotor axis and that this configuration is stable. Although a first successful application of the above attitude acquisition technique was carried out in the RCA Satcom I satellite (December 1975), analytical verification of the intuition has been elusive. However, see the work of Carl Hubert ([7, 8, 9]) for a fundamental effort in this direction.

The difficulty lies in the correct choice of energy function (or Lyapunov function) and the determination of the appropriate invariant manifold for the problem. In the presence of

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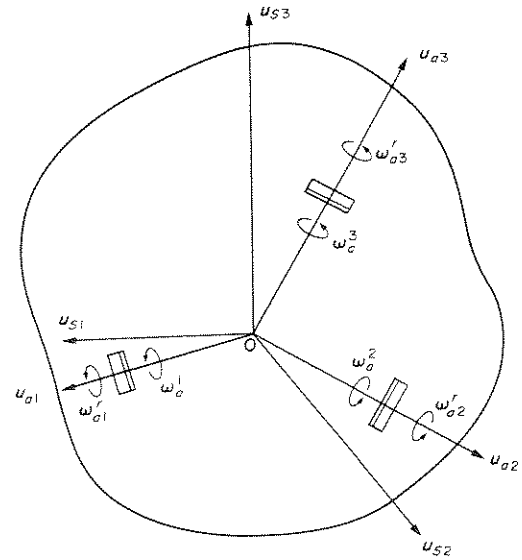

FIG. 2.

damping, we shall see that momentum spheres are replaced by noncompact momentum varieties as invariant manifolds. The choice of energy function is very much dependent on the damping mechanism and the study of critical point structure is complicated by the noncompactness of momentum varieties. In his paper [8], Hubert treated a particular damping mechanism and identified a Lyapunov function, but this study remains incomplete for the lack of critical point information.

In this paper we treat the dynamics of rigid spacecraft carrying three (motor driven or free spinning) rotors (see Fig. 2). Starting from the basic equations in Section 2 we determine in Section 4, the Lie–Poisson structures for the cases of driven rotors and free-spinning rotors. The necessary background material on Lie–Poisson structures is included in Section 3. In Section 4, we also derive in a simple manner the usual design conditions for dual-spin spacecraft by investigating the conditions under which the Hamiltonian of the driven rotor case is a perfect Morse function on the momentum sphere. (Compare with the calculations of Hubert in [9].) Generalizations of these perfectness conditions are contained in [20].

In Section 5, we show that a naive damping mechanism for the driven rotor case with the Hamiltonian as Lyapunov function fails. In Section 6, we treat the case of damped free-spinning rotors and investigate asymptotic behavior. The appendix contains a summary of the results of [20].

Note. Throughout this paper we will assume that each rotor is spinning about an axis of symmetry passing through the center of mass. In the absence of such symmetry the motions can be quite complex (See [21]).

2. BASIC EQUATIONS FOR RIGID SPACECRAFT WITH ROTORS

A standard reference for this section is the paper of Meyer [13]. Consider the rigid spacecraft as a rigid body as in Fig. 2, with rotors attached along the body axes specified by the unit vectors $\{u_{a1}, u_{a2}, u_{a3}\}$ attached to *the body*. We are given an *orthonormal* frame $u_s = (u_{s1}, u_{s2}, u_{s3})$ fixed in inertial space. The attitude of the body relative to the inertial frame u_s is specified by the element A_{as} of $SO(3)$ that carries the inertial frame to the body frame $u_a = (u_{a1}, u_{a2}, u_{a3})$:

$$A_{as}u_s = u_a. \tag{2.1}$$

Rotation of the body causes the element A_{as} of $SO(3)$ to evolve with time and this evolution obeys the basic kinematic equation

$$\dot{A}_{as} = S(\omega_a)A_{as}, \tag{2.2}$$

where, for any $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3$, $S(x)$ denotes the skew-symmetric matrix

$$S(x) = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}. \tag{2.3}$$

The vector ω_a represents the angular velocity of the body relative to the inertial frame. Let $J_a^r = \mathrm{diag}(j_1^r, j_2^r, j_3^r)$, be the diagonal matrix whose element j_i^r denotes the moment of inertia of the ith rotor about its spin axis. Let J_a^v denote the moment of inertia of the body *with the rotors locked* with respect to the body axes. Then the total body angular momentum is given by,

$$h_a = J_a^v\omega_a + J_a^r\omega_a^r. \tag{2.4}$$

Here ω_a^r is the vector of spin velocities of the rotors. The total angular momentum with respect to the inertial frame is given by,

$$h_s = A_{as}^{\prime}h_a. \tag{2.5}$$

From (2.5) it follows that,

$$\dot{A}_{as}h_s + A_{as}\dot{h}_s = J_a^v\dot{\omega}_a + J_a^r\dot{\omega}_a^r. \tag{2.6}$$

Since there are *no external torques* on the system $\dot{h}_s = 0$. Also substituting for $\dot{A}_{as}$ from the kinematic equation (2.2) we obtain,

$$J_a^v\dot{\omega}_a + J_a^r\dot{\omega}_a^r = S(\omega_a)A_{as}h_s. \tag{2.7}$$

Now let

$$-z_a = \begin{pmatrix} -z_{a1} \\ -z_{a2} \\ -z_{a3} \end{pmatrix}$$

denote the vector of internal torques exerted on the rotors. These could be motor torques or damping torques. In either case, from Newton's law we have,

$$J_a^r(\dot{\omega}_a + \dot{\omega}_a^r) = -z_a \tag{2.8}$$

or equivalently

$$J_a^r \dot{\omega}_a = -J_a^r \dot{\omega}_a^r - z_a. \tag{2.9}$$

Subtracting equation (2.9) from (2.7) one gets,

$$(J_a^v - J_a^r)\dot{\omega}_a = S(\omega_a)A_{as}h_s + z_a. \tag{2.10}$$

The complete system of kinematic and dynamic equations for the case of rotors with torques may be summarized (after further substitutions) as follows (*note*: θ_a^r is the vector of rotor angles):

$$\dot{A}_{as} = S(\omega_a)A_{as} \tag{2.11}$$

$$\dot{\theta}_a^r = \omega_a^r \tag{2.12}$$

$$(J_a^v - J_a^r)\dot{\omega}_a = S(\omega_a)[J_a^v\omega_a + J_a^r\omega_a^r] + z_a \tag{2.13}$$

$$J_a^r\dot{\omega}_a^r = -J_a^v(J_a^v - J_a^r)^{-1}z_a - J_a^r(J_a^v - J_a^r)^{-1}S(\omega_a)[J_a^v\omega_a + J_a^r\omega_a^r]. \tag{2.14}$$

As is clear from the above, the dynamics of the rotors and the dynamics of the body are strongly coupled. It is remarkable that matters simplify considerably in two important cases:

Case (a). The torque z_a is such as to maintain the rotors spinning at *constant* angular velocities. From equation (2.8), by setting $\dot{\omega}_a^r = 0$ we obtain,

$$z_a = -J_a^r\dot{\omega}_a. \tag{2.15}$$

The requisite torque law (2.15) is achieved by using motors coupled to *attitude sensors or other devices.* We therefore refer to this case as the *driven rotor case.* In order to avoid confusion, wherever we refer to the driven rotor case we will use the superscript w (for wheel) instead of the superscript r.

Case (b). The torque z_a is a damping torque on the rotors of the form,

$$z_a = \alpha\omega_a^r \tag{2.16}$$

where $\alpha = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$ is a diagonal matrix of nonnegative elements. We refer to this case as the *free-spinning rotor case.* For the purposes of defining the Lie–Poisson structure for this case we will set $\alpha = 0$. Again, to avoid confusion, whenever we refer to the free spinning rotor case, we will use the superscript d instead of the superscript r.

2.1. *Equations for the driven rotor case*

Let $h_v = J_a^v\omega_a$ and $h_w = J_a^w\omega_a^w$. The substitution of (2.15) into (2.13) and (2.14) yields immediately,

$$\dot{h}_v = S(J_a^{v-1}h_v)[h_v + h_w] \tag{2.1.1}$$

$$\dot{h}_w = 0. \tag{2.1.2}$$

In Section (4) we will show that these are precisely in Lie–Poisson form for the Lie algebra $so(3) \oplus R^3$.

2.2. *Equations for the free spinning rotor case*

Define $J \stackrel{\Delta}{=} J_a^v - J_a^d$. Also, let $h_v = J\omega_a$; $h_d = J_a^d(\omega_a + \omega_a^d)$. Making the substitution,

$$z_a = \alpha\omega_a^d$$

in equation (2.8) we get

$$\dot{h}_d = -\alpha\omega_a^d. \tag{2.2.1}$$

Now observe that,

$$\begin{aligned} J_a^v\omega_a + J_a^d\omega_a^d &= (J_a^v - J_a^d)\omega_a + J_a^d(\omega_a + \omega_a^d) \\ &= J\omega_a + J_a^d(\omega_a + \omega_a^d) \\ &= h_v + h_d. \end{aligned} \tag{2.2.2}$$

Substituting this in equation (2.13) we obtain,

$$\dot{h}_v = S(J^{-1}h_v)[h_v + h_d] + \alpha\omega_a^d. \tag{2.2.3}$$

Now

$$\begin{aligned} \alpha\omega_a^d &= \alpha J_a^{d-1}J_a^d\omega_a^d \\ &= \alpha J_a^{d-1}(J_a^d(\omega_a^d + \omega_a) - J_a^d\omega_a) \\ &= \alpha J_a^{d-1}h_d - \alpha\omega_a \\ &= \alpha J_a^{d-1}h_d - \alpha\omega_a \\ &= \alpha J_a^{d-1}h_d - \alpha J^{-1}h_v. \end{aligned} \tag{2.2.4}$$

Substituting from equation (2.2.4) into equations (2.2.1) and (2.2.3), we get the system,

$$\begin{aligned} \dot{h}_v &= S(J^{-1}h_v)[h_v + h_d] - \gamma h_v + \delta h_d \\ \dot{h}_d &= \gamma h_v - \delta h_d \end{aligned} \tag{2.2.5}$$

where the parameters are, $\delta = \alpha(J_a^d)^{-1}$, $\gamma = \alpha(J)^{-1}$, $J = J_a^v - J_a^d$.

The system (2.2.5) determines completely the dynamics of the free spinning rotor case with damping. Secondly, we see that when there is no damping (i.e. $\alpha = 0$ and hence, $\gamma = 0 = \delta$). *the equations* (2.2.5) *are identical in form to equations* (2.1.1) *and* (2.1.2). *This will imply that the Lie–Poisson structure of the free-spinning rotor case with no damping will be identical to that of the driven rotor case.* However the quantity h_v, will have different meanings in these two contexts, because of the different scalings of ω_a. A similar remark applies to h_w and h_d too.

3. LIE–POISSON STRUCTURES: GENERALITIES

First let us recall the following classical fact:
Let R^{2n} be the Cartesian space with coordinates $(q_1, \cdots q_n, p_1, \cdots p_n)$. Let $F(R^{2n})$ denote the space of smooth real-valued functions on R^{2n}. This space can be given the structure of an infinite dimensional Lie algebra over the reals by defining the bracket (known as Poisson bracket)

$$\{f, g\} = \sum_{i,j} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_j}\right), f, g \in F(R^{2n}).$$

This basic fact and its consequences lie at the heart of classical mechanics in R^{2n}. If we wish to understand mechanics in other spaces, it is pertinent to ask to what extent the notion of a Poisson bracket generalizes to other spaces of functions besides $F(R^{2n})$. There are several generalizations available.

(1) (M, ω) is a symplectic manifold, i.e. M is a smooth manifold of dimension $2n$, and ω is a closed, nondegenerate 2-form on M. Thus

$$\omega: \text{Vect}(M) \times \text{Vect}(M) \to F(M)$$

is an $F(M)$ bilinear skew symmetric map where $F(M)$ is the space of smooth real valued functions on M and $\text{Vect}(M)$ is the space of smooth vector fields on M. Nondegeneracy implies that $\omega(X, Y) = 0$, $\forall Y \in \text{Vect}(M) \Rightarrow X = 0$. Further $d\omega = 0$.

The Poisson bracket of two function $f, g \in F(M)$ is defined in the following way: first associate with f, a vector field X_f by requiring that the relation,

$$\omega(X_f, Y) = \mathrm{d}f(Y)$$

hold for all $Y \in \text{Vect}\,(M)$. Then the Poisson bracket is defined as

$$\{\cdot, \cdot\}: F(M) \times F(M) \to F(M) \quad (f, g) \to \{f, g\} = \omega(X_f, X_g).$$

It follows from the properties of ω that $(F(M), \{\cdot, \cdot\})$ is a Lie algebra. The Poisson structure $\{\cdot, \cdot\}$ is said to be nonsingular since it is associated with a nondegenerate form ω.

Odd dimensional manifolds do not admit symplectic structures. To cover this case as well one has the further generalization.

(2). $(M, \tilde{\omega})$ is a *cosymplectic manifold* i.e.

$$\tilde{\omega}: \Omega^1(M) \times \Omega^1(M) \to F(M).$$

is an $F(M)$-bilinear skew symmetric map and $\Omega^1(M)$ = space of 1 forms, and the following condition holds:

$$\text{the bracket } \{\cdot, \cdot\}: F(M) \times F(M) \to F(M), (f_1, f_2) \to \tilde{\omega}(\mathrm{d}f_1, \mathrm{d}f_2),$$

satisfies the Jacobi identity

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0.$$

Clearly every cosymplectic structure gives rise to a Poisson structure on $F(M)$. Cosymplectic structures appear first in the work of Sophus Lie (see the remarks by Hermann in [5]).

The local coordinate representation of a cosymplectic structure is useful. Let $x_1, x_2, \ldots, x_n$

be local coordinates on M and let $dx_1, \ldots, dx_n$ denote the corresponding differential forms. Let,

$$\tilde{\omega}(dx_i, dx_j) = \tilde{\omega}_{ij}(x)\ i, j = 1, 2, \ldots, n.$$

Then the corresponding Poisson bracket is,

$$\{f_1, f_2\} = \sum_i \sum_j \tilde{\omega}_{ij}(x) \frac{\partial f_1}{\partial x_i} \frac{\partial f_2}{\partial x_j}. \tag{3.1}$$

The Jacobi identity imposes additional quadratic conditions on the skew symmetric matrix $(\tilde{\omega}_{ij}(x))$. The rank of a cosymplectic structure at $x \in M$ is just the rank of the matrix $(\tilde{\omega}_{ij}(x))$. A cosymplectic structure is said to be nonsingular if rank $(\tilde{\omega}_{ij}(x)) = \text{constant} = \dim(M) = n$. Nonsingular cosymplectic structures may be dualized to obtain symplectic structures and hence arise only when n is even.

The cosymplectic structure of central interest is the one that is associated *naturally* with Lie algebra $\mathscr{L}$. We discuss this below.

(3). Let $\mathscr{L}$ be a Lie algebra. Let $\mathscr{L}^*$ denote the dual space of linear functionals on $\mathscr{L}$. We think of $\mathscr{L}^*$ as a cosymplectic manifold in the following way:

If $\phi, \psi \in F(\mathscr{L}^*)$ the space of smooth real-valued functions on $\mathscr{L}^*$, we define the Lie–Poisson bracket of ϕ and ψ to be $\{\phi, \psi\}$ satisfying,

$$\{\phi, \psi\}(f) = \langle f, [(D\phi)_f, (D\psi)_f] \rangle. \tag{3.2}$$

Here $f \in \mathscr{L}^*$, $(D\phi)_f$ is the Fréchet derivative of ϕ at f viewed naturally as an element of $\mathscr{L}$, and similarly for $(D\psi)_f$, and $\langle f, \xi \rangle$ denotes the evaluation of the linear functional f at $\xi \in \mathscr{L}$.

It is possible to obtain other Poisson structures on $F(\mathscr{L}^*)$ with respect to special decompositions $\mathscr{L} = a + b$ into subalgebras (see the papers of Reyman and Semenov-Tian-Shansky [15, 16], Ratiu [14], Kupershmidt and Manin [12]). However the structure (3.2) is natural for two reasons both having to do with coadjoint orbits. Since these are relevant to our problems we discuss these below.

A Lie algebra $\mathscr{L}$ acts on its dual $\mathscr{L}^*$ by the coadjoint action ad^*:

$$\begin{aligned} ad^* : \mathscr{L} \times \mathscr{L}^* &\to \mathscr{L}^* \\ (\xi, f) &\to ad_\xi^* f \\ (ad_\xi^* f)(\eta) &= \langle f, [\xi, \eta] \rangle \\ \text{for } \xi, \eta &\in \mathscr{L}. \end{aligned} \tag{3.3}$$

By exponentiating ad^* we obtain an orbit in $\mathscr{L}^*$ passing through f which we denote as $\mathbb{O}_f$. These are known as coadjoint orbits. We have the following facts.

Fact 1. All orbits $\mathbb{O}_f$ are even dimensional and carry a natural symplectic structure ω known as the Kirillov 2-form (since it appeared first in Kirillov's work on the infinite-dimensional representation theory of nilpotent groups; see Kirillov [10] for details). At $f \in \mathbb{O}_f$, $T_f(\mathbb{O}_f)$ the tangent space is isomorphic to $\mathscr{L}/Z_f$ where $Z_f = \{\xi \in \mathscr{L}\colon ad_\xi{}^* f = 0\}$. Letting $[\xi_1], [\xi_2] \in \mathscr{L}$ denote the equivalence classes (tangent vectors) of ξ_1, ξ_2 at f, ω is defined at f as

$$\omega_f(\xi_1, \xi_2) = \langle f, [[\xi_1], [\xi_2]] \rangle. \tag{3.4}$$

Elsewhere ω is given by translation of ω_f.

Fact 2. The natural transitive action of $\mathscr{L}$ on $\mathbb{O}_f$ by vector fields leaves ω invariant. Thus to each $\xi \in \mathscr{L}ad^*$ associates a (globally) Hamiltonian vectorfield on $\mathbb{O}_f$.

Fact 3. The symplectic manifold $(\mathbb{O}_f, \omega)$ has a Poisson structure $\{\cdot, \cdot\}_{f,\omega}$. On the other hand, we can restrict the Lie–Poisson structure $\{\cdot, \cdot\}$ defined by (3.2) to functions on the orbit $\mathbb{O}_f$ and denote this as $\{\cdot, \cdot\}_f$. We have the equality,

$$\{\cdot, \cdot\}_{f,\omega} = \{\cdot, \cdot\}_f. \tag{3.5}$$

This is our primary reason for thinking of $\{\cdot, \cdot\}$ as defined by (3.2) to be natural. See also Kostant's paper [11].

There is a second reason for the naturality of (3.2). Suppose $H \in F(\mathscr{L}^*)$. Consider the Liouville-type equation,

$$\frac{\partial \phi}{\partial t} = \{H, \phi\}. \tag{3.6}$$

Let $\mathfrak{X} := \{X_1, \ldots X_n\}$ be a basis for the Lie algebra $\mathscr{L}$. Let $\{\Gamma^i_{jk}, i, j, k = 1, 2, \ldots, n\}$ denote the set of structure constants for $\mathscr{L}$ in this basis, i.e.

$$[X_i, X_j] = \sum_{k=1}^{n} \Gamma^k_{ij} X_k. \tag{3.7}$$

Let $\mathfrak{F} := \{f_1, f_2, \ldots, f_n\}$ denote the dual basis for $\mathscr{L}^*$, i.e

$$f_i(X_j) = \delta^j_i \text{ the Kronecker symbol.}$$

$\mathscr{L}^*$ has global coordinates $(x_1, \ldots, x_n)$ with respect to the basis $\{f_1, \ldots, f_n\}$. If we let these coordinate function obey the Liouville-type equation (3.6) with respect to some H then we obtain a system of differential equations,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j,l} \Gamma^l_{j,i} \frac{\partial H}{\partial x_j} x_l \qquad i = 1, 2, \ldots n. \tag{3.8}$$

We call this system the Lie–Poisson system associated to the triple $(\mathscr{L}, \mathfrak{X}, H)$ or the triple $(\mathscr{L}^*, \mathfrak{F}, H)$. The vector field on $\mathscr{L}^*$ associated to equation (3.8) is denoted as X_H.

Fact 4. The vector fields X_H defined by (3.8) leave invariant the coadjoint orbits in $\mathscr{L}^*$ and the Kirillov 2-form on an orbit, when restricted to that orbit. This is essentially a consequence of (3.5). What this implies is that Lie–Poisson equations of the form (3.8) define (globally) Hamiltonian systems when restricted to an orbit. The corresponding Hamiltonian is given by the restriction of H to that orbit.

Remarks. For the proofs of results about symplectic manifolds, coadjoint orbits and Poisson structures, see the standard references, Arnold [2], Abraham and Marsden [1], Kirillov [10, chapter 15] and Guillemin and Sternberg [4]. The treatment of singular cosymplectic structures is not so accessible, however see Hermann [5] and the references therein. For Lie–Poisson structures and equations, see the earlier mentioned papers of Reyman and Semenov-Tian-Shansky, Ratiu and the paper by Holmes and Marsden [6]. The last-mentioned is most germane to our work since it treats problems very closely related to our work (including the spinning top in gravity and the rigid body with asymmetric rotor with a *model* Hamiltonian).

Given a system of differential equations how does one recognize it as an equation of the Lie–Poisson type? From our discussions, it is clear that there are three ingredients to this problem.

(i) Identify the correct Lie algebra $\mathscr{L}$ (or its dual $\mathscr{L}^*$).

(ii) Identify the correct basis $\mathfrak{X}$ for $\mathscr{L}$ (or the dual basis $\mathscr{F}$ for $\mathscr{L}^*$).

(iii) Identify the correct Hamiltonian, H (up to a function constant on a coadjoint orbit).

In the next section we carry out these identifications for both the driven rotor case and the free spinning rotor case with no damping. An easy example is given by the standard Euler equations for a rigid body (about principal axes)

$$\begin{aligned} \dot{m}_1 &= m_2 m_3 \frac{(J_2 - J_3)}{J_2 J_3} \\ \dot{m}_2 &= m_3 m_1 \frac{(J_3 - J_1)}{J_1 J_3} \\ \dot{m}_3 &= m_1 m_2 = \frac{(J_1 - J_2)}{J_1 J_2}. \end{aligned} \tag{3.9}$$

Here $\mathscr{L} = so(3)$, the basis $\mathfrak{X}$ for $\mathscr{L}$ is the standard one $\{X_1, X_2, X_3\}$ satisfying $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$ and the Hamiltonian

$$H = \frac{m_1^2}{2J_1} + \frac{m_2^2}{2J_2} + \frac{m_3^2}{2J_3}$$

see Holmes and Marsden [6].

The vector field X_H specified by (3.9) leaves invariant the coadjoint orbits in $so^*(3)$. In the basis dual to $\mathfrak{X}$ these orbits are the spheres

$$m_1^2 + m_2^2 + m_3^2 = \mu^2$$

centred at the origin in $so(3)^*$.

If $\mathscr{L}$ is an abelian Lie algebra its coadjoint orbits are simply points in $\mathscr{L}^*$, since the action (3.3) in this case is trivial. For some more concrete examples of coadjoint orbits see Guillemin and Sternberg [4]. See also the paper of Weinstein for recent results on Poisson structures [23].

4. LIE–POISSON STRUCTURE OF RIGID SPACECRAFT WITH DRIVEN OR FREE-SPINNING ROTORS

In this section we follow the general procedure outlined at the end of Section 3.

Let $\mathscr{L} = so(3) \oplus R^3$ be the direct sum of the Lie algebra of $So(3)$ and the Lie algebra of T^3 the 3-torus. Let $\tilde{\mathfrak{X}} = \{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{L}_1, \tilde{L}_2, \tilde{L}_3\}$ denote the standard basis for $\mathscr{L}$ obeying the commutation rules.

$$\begin{aligned} &[\tilde{M}_1, \tilde{M}_2] = \tilde{M}_3 \\ &[\tilde{M}_2, \tilde{M}_3] = \tilde{M}_1 \\ &[\tilde{M}_3, \tilde{M}_1] = \tilde{M}_2 \\ &[\tilde{L}_i, \tilde{L}_j] = 0 \qquad i, j = 1, 2, 3 \\ &[\tilde{M}_i, \tilde{L}_j] = 0 \qquad i, j = 1, 2, 3. \end{aligned} \tag{4.1}$$

Define a new basis $\mathfrak{X} = \{M_1, M_2, M_3, L_1, L_2, L_3\}$ by setting,

$$
\begin{aligned}
M_i &= \hat{M}_i = \hat{L}_i, \\
L_i &= \hat{L}_i. \qquad i = 1, 2, 3.
\end{aligned}
\tag{4.2}
$$

With respect to this new basis $\mathfrak{X}$, the commutation rules become,

$$
\begin{aligned}
[M_1, M_2] &= [\hat{M}_1 - \hat{L}_1, \hat{M}_2 - \hat{L}_2] \\
&= [\hat{M}_1, \hat{M}_2] \\
&= \hat{M}_3 \\
&= M_3 + L_3, \\
[M_2, M_3] &= M_1 + L_1, \\
[M_3, M_1] &= M_2 + L_2, \\
[L_i, L_j] &= 0, \qquad i, j = 1, 2, 3 \\
[M_i, L_j] &= 0.
\end{aligned}
\tag{4.3}
$$

Denote, $\mathfrak{X} = \{X_1, X_2, \ldots, X_6\}$.
where,

$$
\begin{aligned}
X_i &= M_i \qquad i = 1, 2, 3 \\
X_i &= L_{i-3} \qquad i = 4, 5, 6.
\end{aligned}
$$

Then the structure constants for $\mathcal{L}$ in the $\mathfrak{X}$ basis are,

$$
\begin{aligned}
&\Gamma_{12}^3 = 1; \qquad \Gamma_{12}^6 = 1 \\
&\Gamma_{23}^1 = 1; \qquad \Gamma_{23}^4 = 1 \\
&\Gamma_{31}^2 = 1; \qquad \Gamma_{31}^5 = 1 \\
&\Gamma_{ij}^k = 0 \qquad i, j \in \{4, 5, 6\} \\
&\Gamma_{ij}^k = 0 \qquad i \in \{1, 2, 3\}, j \in \{4, 5, 6\} \\
&\Gamma_{ij}^k = -\Gamma_{ji}^k.
\end{aligned}
$$

Now let $\mathcal{F}$ denote the basis for $\mathcal{L}^*$ which is dual to $\mathfrak{X}$ in the sense of Section 3. Denote the corresponding coordinates for $\mathcal{L}^*$ as

$$
(x_1, x_2, x_3, x_4, x_5, x_6) = (m_1, m_2, m_3, l_1, l_2, l_3)
$$

For any given $H\varepsilon F(\mathcal{L}^*)$, the Lie–Poisson equations (3.8) take the form (with reference to the triple $(\mathcal{L}, \mathfrak{X}, H)$):

$$\dot{m}_1 = \frac{\partial H}{\partial m_3}(m_2 + l_2) - \frac{\partial H}{\partial m_2}(m_3 + l_3)$$

$$\dot{m}_2 = \frac{\partial H}{\partial m_1}(m_3 + l_3) - \frac{\partial H}{\partial m_3}(m_1 + l_1)$$

$$\dot{m}_3 = \frac{\partial H}{\partial m_2}(m_1 + l_1) - \frac{\partial H}{\partial m_1}(m_2 + l_2) \qquad (4.4)$$

$$\dot{l}_1 = 0$$

$$\dot{l}_2 = 0; \quad \dot{l}_3 = 0.$$

4.1. *The driven rotor case*

Referring to Section (2.1) let $M \in O(3)$ be such that

$$M'J_a^v M = \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3);$$

λ_i's are the principal axes moments of inertia. Define $\bar{h}_v = M'h_v$; $\bar{h}_w = M'h_w$. Then equation (2.1.1) and (2.1.2) take the form,

$$\dot{\bar{h}}_v = S(\Lambda^{-1}\bar{h}_v)[\bar{h}_v + \bar{h}_w] \qquad (4.1.1)$$

$$\dot{\bar{h}}_w = 0.$$

More explicitly, if we let,

$$\bar{h}_v = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}; \bar{h}_w = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

then, (4.1.1) reduces to,

$$\dot{m}_1 = \frac{m_3}{\lambda_3}(m_2 + l_2) - \frac{m_2}{\lambda_2}(m_3 + l_3)$$

$$\dot{m}_2 = -\frac{m_3}{\lambda_3}(m_1 + l_1) + \frac{m_1}{\lambda_1}(m_3 + l_3)$$

$$\dot{m}_3 = \frac{m_2}{\lambda_2}(m_1 + l_1) - \frac{m_1}{\lambda_1}(m_2 + l_2) \qquad (4.1.2)$$

$$\dot{l}_1 = 0$$

$$\dot{l}_2 = 0$$

$$\dot{l}_3 = 0.$$

These are *precisely the Lie–Poisson equations* on $\mathscr{L}^* = so^*(3) \oplus R^3$ given by equation (4.4) if we set,

$$\underline{H} \Delta H_w = \frac{m_1^2}{2\lambda_1} + \frac{m_2^2}{2\lambda_2} + \frac{m_3^2}{2\lambda_3}. \tag{4.1.3}$$

Thus H_w is the Hamiltonian. It is obvious that,

$$H_w = \tfrac{1}{2}\langle h_v, (J_a^v)^{-1} h_v \rangle.$$

Now H_w has the following physical interpretation:

Total kinetic energy of the spacecraft

$$\begin{aligned} E_w &= \tfrac{1}{2}\langle \omega_a, J_a^v \omega_a \rangle - \tfrac{1}{2}\langle \omega_a, J_a^w \omega_a \rangle + \tfrac{1}{2}\langle (\omega_a + \omega_a^w), J_a^w(\omega_a + \omega_a^w)\rangle \\ &= \tfrac{1}{2}\langle \omega_a, J_a^v \omega_a \rangle + \tfrac{1}{2}\langle \omega_a^w, J_a^w \omega_a^w \rangle + \langle \omega_a, J_a^w \omega_a^w \rangle \\ &= \tfrac{1}{2}\langle h_v, J_a^{v^{-1}} h_v \rangle + \tfrac{1}{2}\langle h_w, J_a^{w^{-1}} h_w \rangle + \langle \omega_a, J_a^w \omega_a^w \rangle \\ &= H_w + W + Q, \end{aligned}$$

W = wheel kinetic energy relative to platform = constant.

The quantity $Q = \langle \omega_a, J_a^w \omega_a^w \rangle$ has the following interpretation

$$\begin{aligned} \frac{dQ}{dt} &= \langle \omega_a, J_a^w \dot{\omega}_a^w \rangle + \langle \dot{\omega}_a, J_a^w \omega_a^w \rangle \\ &= 0 + \langle J_a^w \dot{\omega}_a, \omega_a^w \rangle \\ &= \langle -z_a, \omega_a^w \rangle \\ &= \text{rate at which motor does work on the rotor.} \end{aligned}$$

Thus, Q is the (conservative) energy *storage function* associated to the supply rate $\langle -z_a, \omega_a^w \rangle$ in the sense of Willems [18]. Thus the Hamiltonian is

$$\begin{aligned} H_w = &\ (\text{total kinetic energy} - \text{storage function}) \\ &- (\text{kinetic energy of } \textit{wheels relative} \text{ to spacecraft}) \end{aligned}$$

The second term on the right is a constant on the trajectories of the Lie–Poisson equations (4.4) and as such is immaterial. It seems that the Hamiltonian H_w is what Hubert calls core energy [8].

From Section 3 we know that the trajectories of the system (4.1.2) leave the coadjoint orbits invariant. What are these orbits?

For $\mathscr{L} = so(3) \oplus R^3$, the orbits in the basis dual to the basis $\tilde{\mathfrak{X}}$ are precisely the products.

$$\{(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3) : \tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2 = \mu^2\} \quad x \quad \{(l_1, l_2, l_3)\} \subset so^*(3) \oplus R^3 \tag{4.1.4}$$

The first factor in (4.1.4) is a sphere of radius μ *centred at the origin* and the second factor is a point. Thus we get a 4-parameter family of 2-dimensional orbits and 3-parameter family of 0-dimensional orbits.

With respect to the basis in $\mathscr{L}^*$ dual to $\mathfrak{X}$, the coadjoint orbits look like the products,

$$\{(m_1, m_2, m_3); (m_1 + l_1)^2 + (m_2 + l_2) + (m_3 + l_3)^2 = \mu^2\} x \{(l_1, l_2, l_3)\}. \tag{4.1.4}$$

Thus the spheres are now centred at the points $(-l_1, -l_2, -l_3)$.

Recall from Section 2, equation (2.5) that,

$$h_s = A_{as}(h_v + h_w).$$

Thus $\|h_s\| = \|h_v + h_w\| = \mu_w$ constant, since h_s is constant and $A_{as} \in SO(3)$. On the other hand,

$$\begin{aligned} \mu_w^2 = \|h_v + h_w\|^2 &= \|M(\bar{h}_v + \bar{h}_w)\|^2 \\ &= \|\bar{h}_v + \bar{h}_w\|^2 \\ &\quad (\text{since } M \in O(3)) \\ &= (m_1 + l_1)^2 + (m_2 + l_2)^2 + (m_3 + l_3)^2. \end{aligned}$$

Thus invariance of coadjoint orbits is a consequence of the conservation of the norm of the total body angular momentum. We call the invariant sphere,

$$(m_i + l_1)^2 + (m_2 + l_2)^2 + (m_3 + l_3)^2 = \mu_w^2$$

the *momentum sphere*. We have the following,

THEOREM 4.1.1. The dynamics of a spacecraft with three driven rotors maintained at constant angular velocities is given by equations (2.1.1) and (2.1.2). The corresponding flow leaves invariant the momentum sphere,

$$\|h_v + h_w\|^2 = \mu_w^2.$$

When restricted to the momentum sphere, equations (2.1.1) and (2.1.2) describe a Hamiltonian system with Hamiltonian,

$$H_w = \langle h_v, J_a^{v-1} h_v \rangle.$$

Remark 4.1.1. The symplectic 2-form on the momentum sphere $\|h_v + h_w\|^2 = \mu_w$ is the pullback of the Kirillov 2-form on the momentum sphere $\|\bar{h}_v + \bar{h}_w\|^2 = \mu^2$ under the mapping $h_v \rightarrow M'\bar{h}_v = h_v$, $h_w \rightarrow M'h_w = \bar{h}_w$. We leave the reader to verify this.

4.2. *The free-spinning rotor case*

Consider the free-spinning rotor equations (2.2.5) with damping parameters γ, δ set to zero:

$$\begin{aligned} \dot{h}_v &= S(J^{-1}h_v)[h_v + h_d] \\ \dot{h}_d &= 0. \end{aligned} \tag{4.2.1}$$

These equations are formally identical to the equations of the driven rotor case. Thus the changes of variables,

$$\bar{h}_v = \bar{M}'h_v;\ \bar{h}_w = \bar{M}'h_w$$

with $M \in O(3)$ satisfying,

$$\bar{M}' J \bar{M} = \bar{\Lambda} = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$$

bring the equations (4.2.1) to the standard Lie–Poisson form (4.4) associated to the triple $(so(3) \oplus R^3, \mathfrak{X}, H)$, where the Hamiltonian,

$$H \underline{\Delta} H_d = \tfrac{1}{2}\langle h_v, J^{-1} h_v \rangle. \tag{4.2.2}$$

We have,

THEOREM 4.2.1. The dynamics of a spacecraft with three free-spinning rotors with no damping is given by (4.2.1). The corresponding flow leaves invariant the moment sphere

$$\|h_v + h_d\|^2 = \mu_d^2.$$

When restricted to the momentum sphere, (4.2.1) defines a Hamiltonian system with the Hamiltonian

$$H_d = \tfrac{1}{2}\langle h_v, J^{-1} h_v \rangle.$$

Remark 4.2.1. It is remarkable that although the driven rotor case and the free-spinning rotor case describe quite different physical situations, they share the same underlying Lie–Poisson structure. Note however, that different moments of inertia enter into these structures.

Remark 4.2.2. The total spacecraft kinetic energy

$$\begin{aligned}
&= \tfrac{1}{2}\langle \omega_a, J_a^v \omega_a \rangle - \tfrac{1}{2}\langle \omega_a, J_a^d \omega_a \rangle + \tfrac{1}{2}\langle (\omega_a + \omega_a^d), J_a^d(\omega_a + \omega_a^d) \rangle \\
&= \tfrac{1}{2}\langle h_v, J^{-1} h_v \rangle + \tfrac{1}{2}\langle h_d, J_a^{d-1} h_d \rangle \\
&= H_d + D.
\end{aligned}$$

Thus the Hamiltonian for the free-spinning rotor case is

$$\begin{aligned}
H_d = {} & \text{(total kinetic energy of spacecraft)} \\
& - \text{(kinetic energy of wheels relative to inertial frame)}.
\end{aligned}$$

It is thus clear that H_d and H_w have quite different physical interpretations.

Remark 4.2.3. The quantities h_w and h_d have quite different physical meanings. Thus $h_w =$ rotor angular momentum relative to body axis and $h_d =$ total body angular momentum − body angular momentum contribution due to the platform alone.

Remark 4.2.4. The quantities μ_w and μ_d are determined by initial conditions alone and are constant due to the absence of external torques.

In the driven rotor case, if we lock the rotors, i.e. $h_w = 0$, then we obtain the classical Euler equations of a rigid body. In the free-spinning rotor case, if we lock the rotors and let $J_d \to 0$ we again get the classical Euler equations. In either case the critical point structure of

$H_w(H_d)$ is well-understood. (See Abraham and Marsden [1, chapter 4, pp. 360–368]). In the present context h_w and h_d play the role of parameters. By spinning up the rotors to desired angular velocities and by maintaining these velocities, h_w can be made to take whatever values we choose (within engineering limitations). If the free-spinning rotors are initially locked (say when the spacecraft enters into orbit) and then released, then $h_d = J_a^d \omega_a^{\text{initial}}$, where $\omega_a^{\text{initial}}$ is the initial spacecraft angular velocity. By choosing $\omega_a^{\text{initial}}$ and J_d appropriatedly we can adjust the parameter h_d. This suggests the problem of the next section.

4.3. *A problem in elementary Morse theory*

Given a constant μ_w (or μ_d) is it possible to choose h_w (or h_d) such that H_w (or H_d) is a *perfect Morse function*, on the momentum sphere $\|h_v + h_w\|^2 = \mu_w^2$ (or $\|h_v + h_d\|^2 = \mu^2$)? Recall that a perfect Morse function on a sphere is a function that has precisely two critical points (which are necessarily a maximum–minimum pair). See the paper of Bott [3] for a beautiful overview of the subject of Morse theory.

Confining ourselves to the driven rotor case, we shall assume first that the body axes are principal axes and the associated moments of inertia are λ_1, λ_2, λ_3 satisfying $\lambda_1 > \lambda_2 > \lambda_3$.

Then our problem takes the form, of finding l_1, l_2, l_3 such that the function

$$H = \frac{m_1^2}{2\lambda_1} + \frac{m_2^2}{2\lambda_2} + \frac{m_3^2}{2\lambda_3}$$

has exactly two critical points on the sphere

$$S_\mu^2\colon (m_1 + l_1)^2 + (m_2 + l_2)^2 + (m_3 + l_3)^2 = \mu^2.$$

(Assume $\mu > 0$.) Recall that, since the Hamiltonian vector field X_H on the momentum sphere determined by the symplectic form ω on the sphere satisfies,

$$\mathrm{d}H(Y) = \omega(X_H, Y),$$

it follows that $\mathrm{d}H = 0$ on the sphere precisely when $X_H = 0$. Thus the critical points of H on S_μ^2 are determined by solving the simultaneous equation,

$$\begin{aligned}
\frac{m_3(m_2 + l_2)}{\lambda_3} &= \frac{m_2(m_3 + l_3)}{\lambda_2}\\
\frac{m_3(m_1 + l_1)}{\lambda_3} &= \frac{m_1(m_3 + l_3)}{\lambda_1}\\
\frac{m_2(m_1 + l_1)}{\lambda_2} &= \frac{m_1(m_2 + l_2)}{\lambda_1}\\
(m_1 + l_1)^2 + (m_2 + l_2)^2 &+ (m_3 + l_3)^2 = \mu^2.
\end{aligned} \tag{4.3.1}$$

Now consider the dual-spin case, i.e.

$l_1 = 0$, $l_2 = 0$, $l_3 = l$. In this case the possible roots of (4.3.1) are listed below:

(i) $m_1 = 0, m_2 = 0, m_3 = \mu - l$

(ii) $m_1 = 0, m_2 = 0, m_3 = -\mu - l$

$$\text{(iii) } m_1 = \sqrt{\mu^2 - \left(\frac{l\lambda_1}{\lambda_1 - \lambda_3}\right)^2},\, m_2 = 0,\, m_3 = \frac{l\lambda_1}{\lambda_1 - \lambda_3} - l$$

$$\text{(iv) } m_1 = -\sqrt{\mu^2 - \left(\frac{l\lambda_1}{\lambda_1 - \lambda_3}\right)^2},\, m_2 = 0,\, m_3 = \frac{l\lambda_1}{\lambda_1 - \lambda_3} - l$$

$$\text{(v) } m_1 = 0,\, m_2 = \sqrt{\mu^2 - \left(\frac{l\lambda_2}{\lambda_2 - \lambda_3}\right)^2},\, m_3 = \frac{l\lambda_2}{\lambda_2 - \lambda_3} - l$$

$$\text{(vi) } m_1 = 0,\, m_2 = -\sqrt{\mu^2 - \left(\frac{l\lambda_2}{\lambda_2 - \lambda_3}\right)^2},\, m_3 = \frac{l\lambda_2}{\lambda_2 - \lambda_3} - l.$$

Suppose that,

$$\frac{l^2}{\mu^2} > \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right)^2. \tag{4.3.2}$$

Then since, $\lambda_1 > \lambda_2 > \lambda_3$ it follows that

$$\frac{l^2}{\mu^2} > \left(\frac{\lambda_2 - \lambda_3}{\lambda_2}\right)^2. \tag{4.3.3}$$

Conditions (4.3.2) and (4.3.3) then force the roots (iii)–(vi) to be nonreal roots and hence these have to be discarded. We have.

THEOREM 4.3.1. Let

$$H = \sum_{i=1}^{3} \frac{m_i^2}{2\lambda_i},\ \lambda_1 > \lambda_2 > \lambda_3.$$

If

$$\left(\frac{l}{\mu}\right)^2 > \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right)^2,$$

then there are precisely two critical points of the function H on the momentum sphere,

$$m_+ = (0, 0, \mu - l)$$

$$m_- = (0, 0, -\mu - l)$$

$$H(m_+) = \frac{(\mu - l)^2}{2\lambda_3};\ H(m_-) = \frac{(\mu + l)^2}{2\lambda_3}.$$

If $l > 0$, m_+ is the minimum and m_- is the maximum. If $l < 0$, m_+ is the maximum and m_- is the minimum.

The conclusions of theorem 4.3.1 appear in [8] in a slightly different form. A comparison with his calculations should convince the reader of the efficacy of the Hamiltonian viewpoint.

Suppose we are again in the dual-spin case but the body axes are not principal. Then the changes of coordinates $h_v \to M'h_v$, $h_w \to M'h_w$ where $M \in O(3)$ satisfies $M'J_a^v M = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > \lambda_2 > \lambda_3$, bring the critical point equations to the form (4.3.1) where,

$$\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = M' \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}.$$

Now suppose that, M' is of the form,

$$M' = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & m_{33} \end{bmatrix}. \tag{4.3.4}$$

Then we are clearly in the context of theorem (4.3.1) and replacing l by $m_{33}l$ we again obtain the conclusions of that theorem for the nonprincipal axes case.

In general the matrix M' does not take the form (4.3.4) and we are forced to investigate the full-fledged 3-parameter problem associated to equation (4.3.1) in order determine the perfectness of H. This study is carried out in [20] the results of which are summarized in the appendix.

The *design condition*

$$\left(\frac{l}{\mu}\right)^2 > \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right)^2$$

for dual-spin spacecraft appears in several places in the aerospace literature. The intuition behind this is the following: Suppose the dual-spin spacecraft carries an additional damping mechanism (many such damping mechanisms including mercury-ring dampers, eddy-current dampers, wheel dampers, etc. are briefly discussed in the book edited by Wertz [17]), then one might expect the dynamics of the spacecraft to follow a trajectory on the momentum sphere which decreases the function H. Such a trajectory will then converge to the unique minimum on the momentum sphere if the design condition

$$\left(\frac{l}{\mu}\right)^2 > \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right)^2$$

holds. This will then complete the dual-spin turn.

The key idea here is that the Hamiltonian H itself be a Lyapunov function in the damped case and that the momentum sphere be invariant. In the next section we show that there is *no linear damping mechanism* which will achieve this.

5. DAMPING: A NEGATIVE RESULT

Consider the Lie–Poisson equations (4.1.2) for the driven rotor case as a system of the form,

$$\dot{m} = f(m)$$
$$\dot{l} = 0. \tag{5.1}$$

We can treat the effect of damping as a perturbation of the equation (5.1) by a vector field of the form $g(m)$ and we write the damped equation as,

$$\dot{m} = f(m) + g(m) \tag{5.2}$$

$$\dot{l} = 0.$$

(a) The condition that the perturbed dynamics leaves the momentum sphere $(m_1 + l_1)^2 + (m_2 + l_2)^2 + (m_3 + l_3)^2 = \mu^2$ invariant requires that the following hold:

$$\langle m + l, g(m) \rangle = 0. \tag{5.3}$$

(b) The condition that H decrease along trajectories of the perturbed dynamics requires that the following hold:

$$\langle m, \Lambda^{-1} g(m) \rangle < 0. \tag{5.4}$$

(c) Now assume that our damping *mechanism* is *linear* i.e.

$$g(m) = \alpha m \tag{5.5}$$

where α is a 3×3 matrix.

Conditions (5.3) and (5.4) take the form:

$$\langle m + l, \alpha m \rangle = 0 \tag{5.3$'$}$$

$$\langle m, \Lambda^{-1} \alpha m \rangle < 0. \tag{5.4$'$}$$

(d) If we further postulate that properties (a) and (b) hold even with the rotors locked ($l = 0$) and for all values of spacecraft momenta m, then (5.3)$'$ and (5.4)$'$ imply that

$$\alpha = -\alpha'$$

$$W \stackrel{\Delta}{=} \Lambda^{-1}\alpha - \alpha\Lambda^{-1} < 0. \tag{5.6}$$

Setting,

$$\alpha = \begin{bmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{bmatrix} \tag{5.7}$$

we find,

$$W = \begin{bmatrix} 0 & \alpha_3\gamma_{12} & -\alpha_2\gamma_{13} \\ \alpha_3\gamma_{12} & 0 & \alpha_1\gamma_{23} \\ -\alpha_2\gamma_{13} & \alpha_1\gamma_{23} & 0 \end{bmatrix} \tag{5.8}$$

where

$$\begin{aligned} \gamma_{12} &= \lambda_1^{-1} - \lambda_2^{-1} \\ \gamma_{23} &= \lambda_2^{-1} - \lambda_3^{-1} \\ \gamma_{13} &= \lambda_1^{-1} - \lambda_3^{-1}. \end{aligned} \tag{5.9}$$

The conditions $\lambda_1 > \lambda_2 > \lambda_3$, imply that $\gamma_{12} < 0$, $\gamma_{23} < 0$, $\gamma_{13} < 0$. With this in mind *we find by appealing to a theorem of Gantmacher* [19, Vol. 1, pp. 308] that any matrix W of the form (5.8) is negative semidefinite iff $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus we conclude that a linear damping mechanism that depends on the spacecraft momenta alone and leaving invariant the momentum sphere with the Hamiltonian playing the role of a Lyapunov function does not exist.

It is clear that a careful study of damping in spacecraft is necessary. In the next section we carry out such a study and prove a stability theorem.

6. SPACECRAFT WITH DAMPED FREE-SPINNING ROTORS AND DRIVEN ROTORS

First recall that if the rigid spacecraft carries only 3 free spinning rotors with damping, then from Section (2.2), the governing equations are,

$$\begin{aligned} \dot{h}_v &= S(J^{-1}h_v)[h_v + h_w] - \gamma h_v + \delta h_d \\ \dot{h}_d &= \gamma h_v - \delta h_d, \end{aligned} \tag{6.1}$$

where $J = J_a^v - J_a^d$, $\delta = \alpha J_a^{d-1}$ and, $\gamma = \alpha J^{-1}$. Recall that,

$$\begin{aligned} h_a &= A_{as} h_s \\ &= J_a^v \omega_a + J_a^d\, \omega_a^d \\ &= (J_a^v - J_a^d)\,\omega_a + J_a^d(\omega_a + \omega_a^d) \\ &= h_v + h_d. \end{aligned} \tag{6.2}$$

Since h_s = constant (no external torques),

$$\|h_v + h_d\| = \text{constant}.$$

Thus the flow of (2.1) leaves invariant the *momentum variety*,

$$M_\mu = \{(h_v, h_d) \colon \|h_v + h_d\|^2 = \mu^2\}. \tag{6.3}$$

M_μ is a sphere-bundle over R^3.

The total kinetic energy of the spacecraft is (see remark 4.2.2),

$$E \underline{\Delta} E(h_v, h_d) \tag{6.4}$$

It is not difficult to check that along trajectories of (6.1),

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= -\langle \omega_a^d, \alpha\omega_a^d \rangle \\ &= -\langle (J^{-1}h_v - J_a^{d-1}h_d), \alpha(J^{-1}h_v - J_a^{d-1}h_d) \rangle. \end{aligned} \tag{6.5}$$

The energy function E is thus a Lyapunov function for the damped free-spinning rotor dynamics. The sublevel sets $E_c = \{(h_v, h_d) \colon E(h_v, h_d) \leqslant c$ and $\|h_v + h_d\|^2 = \mu^2\}$ are positively invariant and compact.

On inspecting equations (6.1), it becomes clear that the equilibrium points are given by the following procedure:

(a) find the roots of the system

$$S(J_a^{v-1}\bar{h}_v) = 0$$
$$\|\bar{h}_v\|^2 = \mu^2; \tag{6.6}$$

(b) then

$$\Sigma = \{(h_v, h_d): h_v = JJ_a^{v-1}\bar{h}_v;\ h_d = J_a^d J_a^{v-1}\bar{h}_v\} \tag{6.7}$$

is the set of equilibrium points.

If the damper moments of inertia are tuned so that J_a^v has distinct eigenvalues then Σ has six isolated points. Further,

$$E_{|\Sigma} = \tfrac{1}{2}\langle \bar{h}_v, J_a^{v-1}\bar{h}_v\rangle. \tag{6.8}$$

Using this formula (or Arnold's formula, see Arnold [1, p. 329, theorem 9], the stable critical points can be identified.

For the purpose of understanding the dual-spin turn it is necessary to consider a spacecraft with two sets of 3 rotors each; one set free-spinning with damping and the other set motor-driven with constant (adjustable) angular velocities. In this case the governing equations are,

$$\begin{aligned}\dot{h}_v &= S(J^{-1}h_v)[h_v + h_d + h_w] - \gamma h_v + \delta h_d\\ \dot{h}_d &= \gamma h_v - \delta h_d\\ \dot{h}_w &= 0,\end{aligned} \tag{6.9}$$

where,

$$h_d = J_a^d(\omega_a + \omega_a^d) \qquad \text{(free rotors)}$$
$$h_w = J_a^w \omega_a^w \qquad \text{(driven rotors)}$$
$$h_v = J\omega_a$$

where, $J = J_a^v - J_a^d$; J_a^v = spacecraft moment of inertia with all rotors locked; $\gamma = \alpha J^{-1}$, $\delta = \alpha J_a^{d-1}$ and α is the diagonal matrix of positive damping coefficients of the free-spinning wheels.

The total kinetic energy of this spacecraft is given by,

$$\begin{aligned}E &\overset{\Delta}{=} E(h_v, h_d)\\ &= \tfrac{1}{2}\langle\omega_a, J_a^v\omega_a\rangle - \tfrac{1}{2}\langle\omega_a, J_a^w\omega_a\rangle + \tfrac{1}{2}\langle(\omega_a + \omega_a^w), J_a^w(\omega_a + \omega_a^w)\rangle\\ &\quad - \tfrac{1}{2}\langle\omega_a, J_a^d\omega_a\rangle + \tfrac{1}{2}\langle(\omega_a + \omega_a^d) J_a^d(\omega_a + \omega_a^d)\rangle\\ &= \tfrac{1}{2}\langle h_v, J^{-1}h_v\rangle + \tfrac{1}{2}\langle h_w, J_a^{w-1}h_w\rangle + \tfrac{1}{2}\langle h_d, J_a^{d-1}h_d\rangle + Q\end{aligned} \tag{6.10}$$

where $Q = \langle\omega, J_a^w\omega_a^w\rangle$. Now the quantity Q is what we call the storage function since,

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \langle J_a^w\dot{\omega}_a, \omega_a^w\rangle\\ &= \langle -z_a, \omega_a^w\rangle\\ &= \text{work done by motor torque } (-z_a).\end{aligned} \tag{6.11}$$

Consider the function,

$$\begin{aligned} V &= E - Q \\ &= \tfrac{1}{2}\langle h_v, J^{-1}h_v\rangle + \tfrac{1}{2}\langle h_w, J_a^{w-1}h_w\rangle + \tfrac{1}{2}\langle h_d, J_a^{d-1}h_d\rangle. \end{aligned} \tag{6.12}$$

It can be verified that along trajectories of (6.9),

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= -\,\langle (J^{-1}h_v - J_a^{d-1}h_d),\ \alpha(J^{-1}h_v - J_a^{d-1}h_d)\rangle \\ &= -\,\langle \omega_a^d,\ \alpha\omega_a^d\rangle. \end{aligned} \tag{6.13}$$

Thus V is a Lyapunov function for (6.9). Now the trajectories of (2.9) leave the momentum variety

$$M_\mu^w := \{(h_v, h_d) \colon \|h_v + h_d + h_w\|^2 = \mu^2\} \tag{6.14}$$

M_μ^w is a 5-dimensional manifold. Observe that $\mathrm{d}V/\mathrm{d}t$ vanishes iff $J^{-1}h_v - J_a^{d-1}h_d = 0$ iff $\gamma h_v = \delta h_d$. Define,

$$\mathcal{R} := \{(h_v, h_d) : (h_v, h_d) \in M_\mu^w \quad \text{and} \quad \frac{\mathrm{d}V}{\mathrm{d}t}(h_v, h_d) = 0\}.$$

It can be checked that,

$$\mathcal{R} = \{(h_v, h_d) : h_v = JJ_a^{v-1}\bar{h}_v;\ h_d = J_a^dJ_a^{v-1}\bar{h}_v \quad \text{and} \quad \|\bar{h}_v + h_w\|^2 = \mu^2\}. \tag{6.15}$$

Thus the set $\mathcal{R}$ has the geometry of a 2-sphere embedded in the momentum variety. Next, note that the set of equilibrium points of (6.9) is given by,

$$\begin{aligned} \Sigma_{\mu,h_w} = \{(h_v, h_d) : h_v = JJ_a^{v-1}\bar{h}_v;\ h_d = J_a^dJ_a^{v-1}\bar{h}_v;\ \|\bar{h}_v + h_w\|^2 \\ = \mu^2;\ S(J_a^{v-1}\bar{h}_v)[\bar{h}_v + h_w] = 0\} \subset \mathcal{R}. \end{aligned} \tag{6.16}$$

From the calculations in Section 4.3 and the more complete results of [20], see also appendix, it is clear that it is always possible to choose h_w (perfectness conditions) such that Σ_{μ,h_w} has precisely two points. For example, in the dual-spin case, with principal axes as driven rotor axes,

$$\begin{aligned} h_w &= (0, 0, l_3)' \\ &= \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \end{aligned} \tag{6.17a}$$

and the perfectness conditions are

$$\left(\frac{l_3}{\mu}\right)^2 > \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right)^2$$

and

$$\left(\frac{l_3}{\mu}\right)^2 > \left(\frac{\lambda_2 - \lambda_3}{\lambda_2}\right)^2. \tag{6.17b}$$

Next we determine the *largest* (positive and negative time) *invariant* manifold M contained in $\mathcal{R}$.

LEMMA 6.1.
$$M = \Sigma_{\mu, h_w}$$

Proof. From (6.9),

$$\gamma \dot{h}_v - \delta \dot{h}_d = \gamma S(J^{-1}h_v)[h_v + h_d + h_w] - \gamma[\gamma h_v - \delta h_d] - \delta[\gamma h_v - \delta h_d].$$

Since, in $\mathcal{R}$, $\gamma h_v = \delta h_d$ and M is invariant, it follows that,

$$\gamma \dot{h}_v - \delta \dot{h}_d = 0 \quad \text{in} \quad M.$$

Hence

$$M = \left\{ \begin{matrix} (h_v, h_d) : S(J^{-1}h_v)[h_v + h_w + h_d] = 0; \ \|h_v + h_d + h_w\|^2 = \mu^2 \\ \gamma h_v - \delta h_d = 0 \end{matrix} \right\}.$$

But this is easily verified to be

$$\Sigma_{\mu, h_w}. \quad \blacksquare$$

Finally note that for any $c \geq 0$, $V^{-1}(c) \subset M_\mu^w$ is positively invariant and compact. We are now ready to apply LaSalle's theorem [22]. Recall its statement:

THEOREM [LaSalle]. (See [22, p. 58].) Let $V(x)$ designate a scalar function with continuous partial derivatives. Let Ω_c designate the region where $V(x) \leq c$. Assume that Ω_c is bounded and that within Ω_c:

$$V(x) > 0 \quad \text{for} \quad x \neq 0$$
$$\dot{V}(x) \leq 0$$

along trajectories of the equation $\dot{x} = X(x)$. Let $\mathcal{R}$ be the set of points within Ω_c where $\dot{V}(x) = 0$, and let M denote the largest invariant set in $\mathcal{R}$. Then every solution $x(t)$ in Ω_c tends to M as $t \to \infty$.

In the present context,

$$\Omega_c = V^{-1}(c) \cap M_\mu^w$$

$\mathcal{R}$ and M are as above. From the conclusions of lemma 6.1, we have the corollary,

COROLLARY. All solutions of (6.9) converge to one of the equilibrium points in

$$\Sigma_{\mu, h_w}.$$

Now with the additional hypothesis of perfectness (such as for example (6.17)), one of the two equilibrium points is a global minimum for V on M_μ^w and the other equilibrium point has an unstable manifold of dimension 2. Only a thin set (contained in a codimension 2 manifold) of trajectories converges to the latter. Thus we can conclude

THEOREM 6.1. In the dual-spin case with perfectness, all but a thin set of trajectories converge to the global minimum of V on M_μ.

7. CONCLUSIONS

This paper was motivated by the problem of analytical verification of certain design conditions for dual-spin spacecraft known to aerospace engineers. To this end, we have carried out a study of the dynamics of spacecraft with driven and free-spinning rotors. We have determined the underlying Lie–Poisson structures for such spacecraft. The Hamiltonian viewpoint is particularly useful in obtaining and generalizing in a very complete way the above-mentioned design conditions (see Appendix and [20]). However the damped case is much more subtle than simply finding conditions for perfectness of the Hamiltonian (see Section 5). We have proved in Section 6 (for the first time) an asymptotic stability theorem for the dual-spin maneuver in the presence of a suitable damping mechanism.

We suspect that the theory of Lie–Poisson structures is a natural tool for the analysis of more complex multi-body spacecraft than the ones treated in this paper.

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APPENDIX (BASED ON JOINT WORK WITH C. A. BERENSTEIN)

In this appendix we summarize the perfectness conditions of [20]. The conditions given below generalize the results of Section 4.3 to the case of nonprincipal axes and three driven rotors.

To fix notation, let J_a^v, h_w, h_v be as before. Since J_a^v is symmetric positive definite, let $M \in O(3)$ be such that

$$M' J_a^v M = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3).$$

Let, $h = h_v + h_w$; $\bar{h}_w = M' h_w$ and $\bar{h} = M'h$ and denote $\bar{h}_w = (q_1, q_2, q_3)'$ and $\bar{h} = (p_1, p_2, p_3)'$.

The equilibrium equations for a spacecraft with three driven rotors then take the form,

$$\begin{aligned} (\lambda_1^{-1} - \lambda)p_1 &= \lambda_1^{-1} q_1 \\ (\lambda_2^{-1} - \lambda)p_2 &= \lambda_2^{-1} q_2 \\ (\lambda_3^{-1} - \lambda)p_3 &= \lambda_3^{-1} q_3 \\ p_1^2 + p_2^2 + p_3^2 &= \mu^2; \mu > 0 \end{aligned} \tag{A.1}$$

and $\lambda \in R$.

The conditions for equation (A.1) to have precisely two solutions (perfectness conditions) are as follows

Case	Rotor momenta	Perfectness conditions
1. λ_i all distinct	$q_1 = 0, \quad q_2 = 0$ $q_3 \neq 0$	$\left(\frac{q_3}{\mu}\right)^2 \geqslant \left(1 - \frac{\lambda_3}{\lambda_1}\right)^2$ and $\left(\frac{q_3}{\mu}\right)^2 \geqslant \left(1 - \frac{\lambda_3}{\lambda_2}\right)^2$
	$q_1 = 0, \quad q_2 \neq 0$ $q_3 \neq 0$	$\mu^2 < \phi_1(\lambda_0)$† *and* $\mu^2 < \left(\frac{\lambda_1 q_3}{\lambda_1 - \lambda_3}\right)^2 + \left(\frac{\lambda_1 q_2}{\lambda_1 - \lambda_2}\right)^2$
	$q_1 \neq 0, \quad q_2 \neq 0$ $q_3 \neq 0$	$\mu^2 < \min(\phi_2(\lambda_0^+), \phi_2(\lambda_0^-))$‡
2. $\lambda_1 = \lambda_2 = \lambda^*$ $\lambda_3 \neq \lambda^*$	$q_1 = 0, \quad q_2 = 0$ $q_3 \neq 0$	$\left(\frac{q_3}{\mu}\right)^2 > \left(\frac{\lambda^* - \lambda_3}{\lambda^*}\right)^2$
	$q_1 = 0, \quad q_2 \neq 0$ $q_3 \neq 0$	$\mu^2 < \phi_3(\lambda_0)$§ *and* $\left(\frac{q_3}{\mu}\right)^2 > \left(\frac{\lambda^* - \lambda_3}{\lambda^*}\right)^2$
	$q_1 \neq 0, \quad q_2 \neq 0$ $q_3 \neq 0$	$\mu^2 < \phi_4(\lambda_0)$‖ *and* $\left(\frac{q_3}{\mu}\right)^2 > \left(\frac{\lambda^* - \lambda_3}{\mu^*}\right)^2$ *and* $\mu^2 < \frac{\lambda_3^2(q_1^2 + q_2^2)}{(\lambda^* - \lambda_3)^2}$
3. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda^*$	In this case perfectness is achieved as long as one of the q_i's is nonzero.	

† Here $\phi_1(\lambda) = \dfrac{q_2^2}{(1-\lambda\lambda_2)^2} + \dfrac{q_3^2}{(1-\lambda\lambda_3)^2}$ and $\lambda_0 = \dfrac{\lambda_3(q_2^2\lambda_2)^{1/3} + \lambda_2(q_3^2\lambda_3)^{1/3}}{(q_2^2\lambda_2)^{1/3} + (q_3^2\lambda_3)^{1/3}}$.

‡ Here $\phi_2(\lambda) = \sum_{i=1}^{3} \dfrac{q_i^2}{(1-\lambda\lambda_i)^2}$, and λ_0^+ and λ_0^- are the local minima of the function ϕ_2 on the real line contained in the interval $(\min(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}), \max(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}))$.

§ Here $\phi_3(\lambda) = \dfrac{q_3^2}{(1-\lambda\lambda_3)^2} + \dfrac{q_2^2}{(1-\lambda\lambda_*)^2}$, and $\lambda_0 = \dfrac{\lambda_3(q_2^2\lambda_*)^{1/3} + \lambda_*(q_3^2\lambda_3)^{1/3}}{(q_2^2\lambda_*)^{1/3} + (q_3^2\lambda_3)^{1/3}}$

‖ Here $\phi_4(\lambda) = \dfrac{q_3^2}{(1-\lambda\lambda_3)^2} + \dfrac{q_2^2}{(1-\lambda\lambda_*)^2}$ and $\lambda_0 = \dfrac{\lambda_3((q_1^2 + q_2)\lambda_*)^{1/3} + \lambda_*(q_3^2\lambda_3)^{1/3}}{((q_1^2 + q_2^2)\lambda_*)^{1/3} + (q_3^2\lambda_3)^{1/3}}$.

For a given μ(= norm of total angular momentum of the spacecraft), it is possible to achieve the perfectness conditions above by choosing q_i large enough in absolute value. This can always be accomplished by spinning up the rotors to high enough angular velocities relative to the spacecraft. This maneuver will not alter μ, since there are no external torques. When the perfectness conditions hold, the resulting equilibria form a maximum minimum pair for the Hamiltonian. In the event that there is additional damping, we show in Section 6 that one of these equilibria is stable and is the global minimum for an appropriate Lyapunov function.

Note. The author would like to thank the referee for pointing out to him the paper by D. D. Holm, J. E. Marsden, T. Ratiu and A. Weinstein, Nonlinear stability conditions and *a priori* estimates for barotropic hydrodynamics, *Phys. Lett.* **98A**, No. 1, 2, 3 (1983). This reference contains a useful technique for stability analysis using the Casimir functions.

Also, after submitting this paper, the author came across the book *Dynamics of Systems of Rigid Bodies* by Jens Wittenburg (B. G. Teubner, Stuttgart, (1977)) which among other things contains a discussion of gyrostats from a different point of view.

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