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Controllability of a class of underactuated mechanical systems with symmetry☆

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Abstract

In this paper we develop results based on geometric mechanics to study the controllability of a class of controlled under-actuated left invariant mechanical systems on Lie groups. We exploit the invariance of the controlled nonlinear dynamics to the group action (symmetry) to derive a set of reduced dynamics for the system. We first present sufficient conditions for the controllability of the reduced dynamics. We prove conditions (boundedness of coadjoint orbits and existence of a radially unbounded Lyapunov function) under which the drift vector field (of the reduced system) is weakly positively Poisson stable (WPPS). The WPPS nature of the drift vector field along with the Lie algebra rank condition is used to show controllability of the reduced system. We then extend these results to the unreduced dynamics, considering separately the cases when the symmetry group is compact and noncompact. In the noncompact case, under further assumptions of equilibrium controllability, sufficient conditions for controllability of the unreduced dynamics are derived. Jetpuck (hovercraft) and underwater vehicles are used as mechanical systems to motivate our work and illustrate our theoretical results.

Keywords: Controllability; Underactuated systems; Symmetry; Lie–Poisson reduction; Jetpuck; Underwater vehicles

1. Introduction

The impact of Lie theory in the context of nonlinear control became prominent around the early 1970s. The fundamental observation that *almost* all the information in the Lie group is contained in its Lie algebra, and questions about systems evolving on Lie groups could be reduced to their Lie algebras, is the cornerstone of the applications of Lie algebras and Lie groups to control theory. In the early 1970s, Brockett, Jurjdevic, Sussmann and others exploited this observation and introduced the theory of Lie groups and their associated Lie algebras into the context of nonlinear control to express notions such as controllability, observability and realization theory for right-invariant systems. One of the most notable applications of Lie-theoretic techniques in control theory has been in determining controllability of nonlinear systems. Results in this area have inspired many interesting approaches in the design of constructive control laws to steer and stabilize nonlinear control systems.

Some of the early works (Lee & Markus, 1976 and references therein) on nonlinear controllability was based on linearization of nonlinear systems. It was observed that if the linearization of a nonlinear system at an equilibrium point is controllable, the system itself is locally controllable. Later, a differential geometric approach to the problem was adopted in which a control system was viewed as a family of vector fields. It was observed that (Hermann, 1968, Chapter 18; Hermes, 1974; Krener, 1974; Lobry, 1970; Sussmann & Jurdjevic, 1972) a lot of the interesting control-theoretic information was contained in the Lie brackets of these vector fields. It was realized (Hermann & Krener, 1977; Krener, 1974) that Chow's theorem (Hermann, 1968, Chapter 18) leads to the characterization of controllability for systems without drift. Chow's theorem provides a Lie algebra rank test, for controllability of nonlinear systems without drift, similar in spirit to that of Kalman's rank condition for linear systems. In the setting of controlled mechanical systems, the Lagrangian dynamics, being second order, necessarily

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include drift. In this setting Chow's theorem cannot be used to conclude controllability. Studying controllability of general systems with drift is usually a hard problem. Important contributions in this direction have been due to Bonnard, Lobry, Crouch, Byrnes, Jurdjevic and Kupka (Jurdjevic & Kupka, 1981), and others. In Crouch and Byrnes (1986) sufficient conditions are given, in terms of a "group action", that a locally accessible system is also locally reachable. In Lobry (1974), sufficient conditions for the controllability of a conservative dynamical nonlinear affine control system on a compact Riemannian manifold are presented. More recently, this result was extended in Lian, Wang, and Fu (1994) to systems where the drift vector field was assumed to be weakly positively Poisson stable. Other contributions include the work of Bullo, Leonard, and Lewis (2000) and Lewis and Murray (1996) on configuration controllability and motion control algorithms, where controllability properties are characterized by algebraic operations of the symmetric product and Lie brackets, and Ostrowski and Burdick (1997) on controllability tests for a class of mechanical systems with symmetries and constraints.

Motivated in part by Lian et al. (1994) and Lobry (1974), the focus of this paper is to study the controllability of a class of underactuated control-affine nonlinear controlled mechanical systems. In particular, we are interested in mechanical systems with configuration space identified with a finite-dimensional Lie group, G, and Hamiltonian dynamics on the cotangent bundle T^*G. We further assume that the affine-controlled nonlinear dynamics are G-invariant.

To infer controllability properties of the Hamiltonian dynamics defined on T^*G, we begin our analysis by studying the reduced dynamics on the quotient space T^*G/G and then extend these results to the unreduced dynamics. The reduction (projection) of the dynamics to the quotient space follows from G-invariance and the Lie–Poisson reduction theorem (Krishnaprasad, 1993; Marsden & Ratiu, 1994; Weinstein, 1983,1985). The fact that the quotient space also inherits a noncanonical Poisson structure (universally known as the Lie–Poisson structure) and the reduced dynamics are Hamiltonian with respect to this bracket, plays a key role in our approach. We show (Theorem 4.3) that depending on the existence of a radially unbounded Lyapunov function, the drift vector field (of the reduced system) is weakly positively Poisson stable. The weak positive Poisson stability of the drift vector field along with the Lie algebra rank condition (Lian et al., 1994) is used to show controllability of the reduced system.

Exploiting controllability of the reduced dynamics, we then present sufficient conditions for controllability of the unreduced dynamics, depending on whether the symmetry group is compact or noncompact. In the setting where the symmetry group is compact we show that under assumptions of Theorem 4.3, we can conclude that the drift vector field on T^*G is also weakly positively Poisson stable. This again enables us to conclude controllability on T^*G. In the setting where the symmetry group is noncompact, under assumptions of equilibrium controllability, we show that we can conclude controllability of the unreduced dynamics. The proof relies on that of Theorem 4.1 and earlier work by Lewis and Murray (1996) on configuration controllability. Our result gives a manageable tool to check for controllability of a wide class of mechanical systems with symmetry. Some of these ideas were presented in Krishnaprasad and Manikonda (1998) and Manikonda and Krishnaprasad (1997).

In practice, a large class of controlled mechanical systems exhibit G-invariant dynamics. Examples of such systems include the jetpuck (or hovercraft), spacecraft and underwater vehicles. In this paper, we use the jetpuck and the underwater vehicle as examples of mechanical systems with G-invariant dynamics to motivate our work.

The paper is organized as follows. In Section 2, we present some mathematical preliminaries followed by a brief discussion on Lie–Poisson reduction. In Section 3, examples of left-invariant mechanical systems are presented and the reduction procedure is outlined. In Section 4 we present our main results on controllability. In Section 4.1, the concept of Poisson stability is introduced and sufficient conditions for controllability of the reduced dynamics are derived. In Section 4.2 these results are extended to the unreduced dynamics. Compact and noncompact symmetry groups are addressed separately. In Section 4.3 relevance of our results to small-time local controllability are presented. These results are used to make appropriate conclusions on the controllability of the mechanical systems discussed in Section 3. Conclusions are presented in Section 5.

2. Mathematical preliminaries

In this section we present, briefly, the geometric framework used in this paper. The reader is referred to Abraham and Marsden (1977), Arnold (1989), Marsden and Ratiu (1994) and Olver (1993) for further details. For the class of mechanical systems discussed, we assume that the configuration space of these systems can be identified with a Lie group G. We model the dynamics of these systems as controlled Hamiltonian systems on T^*G. Written in the form of an affine nonlinear control system, the dynamics take the form

$$\Sigma\colon\ \dot{x} = X_H(x) + \sum_{i=1}^{m} Y_i(x)u_i, \quad x \in T^*G, \ u_i \in \mathscr{U}, \tag{1}$$

where $\mathscr{U}$ is the set of admissible controls, X_H is a Hamiltonian vector field with respect to a Hamiltonian $H : T^*G \to \mathbb{R}$ and the canonical Poisson bracket on T^*G, i.e.

$$X_H(F) = \{F, H\} \quad \text{for every } F \in C^\infty(T^*G) \tag{2}$$

and Y_i, $i = 1, \ldots, m$ are input (control) vector fields defined on T^*G. We further assume that the Hamiltonian H and the control vector fields Y_i are G-invariant, i.e. $H \circ T^*L_g(x) = H(x)$ and $T(T^*L_g)Y_i(x) = Y_i(T^*L_g x)$, $\forall g \in G, x \in T^*G$.

Here L_g denotes the left action of G on itself and T^*L_g the cotangent lift of L_g. Observing that T^*L_g is a Poisson map it follows that since H is G-invariant, X_H too is G-invariant.

Given a G-invariant vector field X defined on a manifold M, if the action of G is free and proper, then there is an induced vector field $\tilde{X}(\pi(x)) = T\pi(X(x))$ on the quotient manifold M/G such that

$$\phi_t^{\tilde{X}}(\pi(x)) = \pi \circ \phi_t^X(x), \tag{3}$$

where $\pi : M \to M/G$ is the projection map and $\phi_t^X(\cdot)$ denotes the flow of the vector field X. While in the general setting solving for $\tilde{X}$ can be quite complicated, in the setting of left-invariant Hamiltonian vector fields defined on Poisson manifolds the geometry can be exploited to solve for $\tilde{X}$.

Recall (Marsden & Ratiu, 1994; Olver, 1993) that if the action $\Phi_g : M \to M$ of a Lie group G on a Poisson manifold $(M, \{\cdot,\cdot\})$ is free and proper, and if Φ_g is a Poisson map, then there exists a unique Poisson structure on M/G denoted by $\{\cdot,\cdot\}_{M/G}$ such that the projection $\pi : M \to M/G$ is a Poisson map. Hence if H is a G-invariant Hamiltonian on M, it defines a corresponding function $\tilde{H}$ on M/G such that $H = \tilde{H} \circ \pi$. Further $T\pi \circ X_H = X_{\tilde{H}} \circ \pi$, i.e. a G-invariant Hamiltonian vector field X_H reduces to the Hamiltonian vector field $X_{\tilde{H}}$ on M/G and $X_{\tilde{H}}$ is Hamiltonian with respect to the reduced Hamiltonian $\tilde{H}$ and the Poisson structure $\{\cdot,\cdot\}_{M/G}$.

Let $\mathfrak{G}$ denote the Lie algebra of G and $\mathfrak{G}^*$ the dual of $\mathfrak{G}$. In the special case where $M = T^*G$ and $M/G = T^*G/G \cong \mathfrak{G}^*$, the Lie–Poisson reduction theorem (Marsden & Ratiu, 1994; Weinstein, 1983,1985; Krishnaprasad, 1993) relates the canonical Poisson bracket on T^*G to the Lie–Poisson bracket on $\mathfrak{G}^*$.

Theorem 2.1. (Lie–Poisson reduction theorem, Marsden & Ratiu, 1994; Weinstein, 1983,1985; Krishnaprasad, 1993). *Identifying the set of functions on $\mathfrak{G}^*$ with the set of left invariant functions on T^*G endows $\mathfrak{G}^*$ with a Poisson structure given by*

$$\{F, G\}_-(\mu) = -\langle \mu, [\nabla F, \nabla H] \rangle \tag{4}$$

for all $F, G \in C^\infty(T^*G)$ and $\mu \in \mathfrak{G}^*$.

The Poisson bracket defined in (4) is called the minus Lie–Poisson bracket. The space $\mathfrak{G}^*$ with this (minus) Poisson structure is denoted by $\mathfrak{G}^*_-$ and the Poisson map $\pi : T^*G \to \mathfrak{G}_-$ is given by

$$(g, \alpha_g) \mapsto T_e^* L_g \cdot \alpha_g, \quad g \in G, \ \alpha_g \in T^*G. \tag{5}$$

Hence, in the current setting, since the manifold of interest to us is T^*G, X_H projects to $X_{\tilde{H}}$ on $T^*G/G \cong \mathfrak{G}^*$ and $X_{\tilde{H}}$ is Hamiltonian on $\mathfrak{G}^*_-$ w.r.t. to the reduced Hamiltonian $\tilde{H}$ and the minus Lie–Poisson bracket defined on $\mathfrak{G}^*$. Further, since $Y_i,\ i = 1, \ldots, m$ are G-invariant, they also project to vector fields $\tilde{Y}_i,\ i = 1, \ldots, m$ on $\mathfrak{G}^*$. Hence, for the class of systems discussed in this paper the dynamics Σ, projects to a set of reduced dynamics on $\mathfrak{G}^*$ given by

$$\tilde{\Sigma} : \dot{\mu} = X_{\tilde{H}}(\mu) + \sum_{i=1}^m \tilde{Y}_i(\mu) u_i, \quad \mu \in \mathfrak{G}^*. \tag{6}$$

The description of the Lie–Poisson bracket in (4) is given in its coordinate free form. Often it is useful to write out the Poisson bracket explicitly in terms of local coordinates defined on $\mathfrak{G}^*$. Given a basis $\{\xi_1, \ldots, \xi_r\}$ for the Lie algebra $\mathfrak{G}$ and the dual basis $\{\xi_1^\flat, \ldots, \xi_r^\flat\}$ for the Lie algebra $\mathfrak{G}^*$ determined by the condition $\langle \xi_i^\flat, \xi_j \rangle = \delta_j^i$, any $\mu \in \mathfrak{G}^*_-$ can be expressed as $\mu = \sum_{i=1}^r \mu_i \xi_i^\flat$. In these local coordinates the minus Lie–Poisson bracket of two differentiable functions $F, G \in C^\infty(\mathfrak{G}^*)$ is then given by

$$\{F, G\}_-(\mu) = -\sum_{i,j,k=1}^r c_{ij}^k \mu_k \frac{\partial F}{\partial \mu_i} \frac{\partial G}{\partial \mu_j}, \tag{7}$$

where $c_{ij}^k,\ i,\ j,\ k = 1, \ldots, r$ are the structure constants of $\mathfrak{G}$ relative to the basis $\{\xi_1, \ldots, \xi_r\}$. Equivalently (7) can be written as

$$\{F, H\}_-(\mu) = \nabla F^{\mathrm{T}} \Lambda(\mu) \nabla H, \tag{8}$$

where

$$[\Lambda(\mu)]_{ij} = -\sum_{k=1}^r c_{ij}^k \mu_k \tag{9}$$

denotes the associated Poisson tensor.

It is also useful to note that the Poisson tensor on $\mathfrak{G}^*_-$ induces a symplectic foliation which has a particularly nice interpretation in terms of the dual to the adjoint representation of the underlying Lie group G on the Lie algebra $\mathfrak{G}$. This is given by the following theorem, which appears to be due to Kirillov, Arnold, Kostant and Souriau, though similar ideas first appear in the work of Lie, Berezin and Weil. (See Marsden & Ratiu, 1994 for historical comments and references.)

Theorem 2.2 (Marsden & Ratiu, 1994). *Let G be a connected Lie group with coadjoint representation Ad^*G on $\mathfrak{G}^*$. Then the orbits of Ad^*G are immersed submanifolds of $\mathfrak{G}^*$ and are precisely the leaves of the symplectic foliation induced by the Lie–Poisson bracket on $\mathfrak{G}^*$. Moreover, for each $g \in G$, the coadjoint map Ad_g^* is a Poisson mapping on $\mathfrak{G}^*$ preserving the leaves of the foliation.*

As we shall see, in the reduced dynamics, the Lie–Poisson structure induced on $\mathfrak{G}^*$ and the geometry on the coadjoint orbits will play an important role in determining controllability properties of the reduced and unreduced system.

3. Examples

In this section we outline, in some detail, the reduction procedure for two examples of mechanical systems with symmetry—the jetpuck and the underwater vehicle (Fig. 1). In addition to providing a better insight into the reduction procedure, these mechanical systems will serve as examples for the application of our main results on controllability that are presented in later sections.

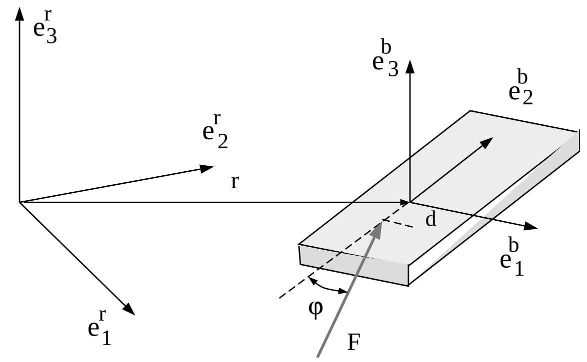

Fig. 1. Jet puck.

3.1. Jetpuck: Planar rigid body with a vectored thrust

We model a jetpuck as a planar rigid body with a vectored thrust (Manikonda & Krishnaprasad, 1996) and identify its configuration space with $SE(2)$.[1] It is further assumed that the line of action of the force is fixed and does not pass through the center of mass. Let Ω denote the angular velocity, $v=(v_1,v_2)$ denote the components of the linear translational velocity along the body fixed frame, m denote the mass, and I denote the moment of inertia (tensor) defined in body-fixed coordinates. Letting $\Omega = R^{-1}\dot{R}$ the Lagrangian $L : TSE(2) \to \mathbb{R}$ is then simply the kinetic energy, i.e.

$$L(R,r,\dot{R},\dot{r}) = \frac{1}{2} I\Omega^2 + \frac{m}{2}\|\dot{r}\|^2. \tag{10}$$

The corresponding Hamiltonian on $T^*SE(2)$ is given by

$$H = \frac{1}{2I}\Pi^2 + \frac{\|p\|^2}{2m}, \tag{11}$$

where $\Pi = I\Omega$ and $p = mv$ denote the angular momentum and linear momentum, respectively. Let L_g denote the left action of $SE(2)$ on itself. Hence given $\bar{g} = (\bar{R},\bar{r}), L_{\bar{g}}g = \bar{g}\cdot g = (\bar{R}R, \bar{R}r + \bar{r})$. Since

$$(TL_{(\bar{R},\bar{r})})L(R,r,\dot{R},\dot{r}) = L(\bar{R}R, \bar{R}r+\bar{r}, \bar{R}\dot{R}, \bar{R}\dot{r}). \tag{12}$$

the Lagrangian is $SE(2)$-invariant and hence the Hamiltonian (11) is also $SE(2)$-invariant.

As the actuation (forces and torques) on the jetpuck are due to body-fixed thrusters/actuators, these forces are invariant to translations and rotations in the plane i.e. invariant to the left action of $SE(2)$, or any subgroup of it. Since in addition the Hamiltonian defined on $T^*SE(2)$ is $SE(2)$-invariant, the Hamiltonian control system is $SE(2)$-invariant.

Hence dynamics on $T^*SE(2)$ projects to Lie–Poisson reduced dynamics on $\mathfrak{G}^* = \mathfrak{se}(2)^*$. The projection $\lambda : T^*G \to \mathfrak{G}^*$ is given by $\lambda : \alpha_g \mapsto (TL_g)^*\alpha_g$. Choosing convected (body) variables $P = R^{\mathrm{T}}p$ and Π as coordinates for $\mathfrak{G}^*$, the reduced Hamiltonian $\tilde{H}$ is given by

$$\tilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m}. \tag{13}$$

The Lie–Poisson tensor on $se(2)^*$ is given by

$$\Lambda = \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}.$$

The reduced Hamiltonian system $\Sigma_{\tilde{H}}$ takes the form

$$\dot{\mu} = X_{\tilde{H}}(\mu) + \tilde{g}u, \tag{14}$$

where $\mu = (P_1, P_2, \Pi)^{\mathrm{T}}, X_{\tilde{H}} = \Lambda(\mu)\nabla\tilde{H}$ and $\tilde{g}u$ is the external force projected appropriately. In the present setting

$$\tilde{g} = (\cos\phi, \sin\phi, |d|\sin\phi)^{\mathrm{T}}.$$

In the rest of the discussions we assume that the line of action of the force is fixed (i.e. ϕ is fixed) but its direction can be reversed. The reduced dynamics defined on $\mathfrak{se}(2)^*$ are given by

$$\begin{aligned} \dot{P}_1 &= P_2\Pi/I + \alpha u, \\ \dot{P}_2 &= -P_1\Pi/I + \beta u, \\ \dot{\Pi} &= d\beta u, \end{aligned} \tag{15}$$

where $\alpha = \cos\phi, \beta = \sin\phi$ and $u \in [1,-1]$. The reduced dynamics combined with the kinematics

$$\begin{aligned} \dot{r} &= \tfrac{RP}{m} \\ \dot{R} &= \frac{R\hat{\Pi}}{I} \end{aligned} \quad \text{where } \hat{\Pi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\Pi \tag{16}$$

define the dynamics on $T^*SE(2) = \mathfrak{se}(2)^* \times SE(2)$. We refer to the dynamics defined on $T^*SE(2)$ as the *jetpuck dynamics* and refer to (15) as the *reduced jetpuck dynamics*.

3.2. Autonomous underwater vehicle

An autonomous underwater vehicle can be modeled as a rigid body submerged in an infinitely large volume of incompressible, inviscid and irrotational fluid which is at rest at infinity (Birkhoff, 1960; Lamb, 1945; Leonard, 1997). We consider an underwater vehicle with ellipsoidal geometry. It can be shown that the impulse of the body–fluid system varies, as a consequence of extraneous forces acting on the solid, in exactly the same way as the momentum of a finite dynamical system. In the case of coincident center of mass and center of gravity (CG), these equations were derived by Lamb (1945) and by Birkhoff in the Lie group setting as early as 1943 (Birkhoff, 1960). The observation that the reduced dynamics for noncoincident center of mass and buoyancy are of the "Lie–Poisson" type was made in Leonard (1997). The reduction procedure discussed in Leonard (1997) is briefly outlined here. This system has a sufficient amount of complexity and serves as a challenging example for application of controllability results derived in later sections.

[1] In the rest of the discussion an element of $SE(n)$, $n = 2,3$ is given by the pair (R,r) where $R \in SO(n)$ and $r \in \mathbb{R}^n$ is a vector from the origin of the inertial frame to the origin of the body frame.

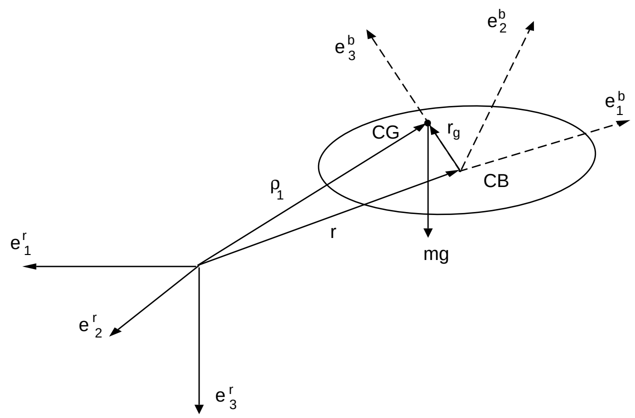

Fig. 2. Underwater vehicle.

Assume that the CG does not coincide with the center of buoyancy (CB) and lies on the e_3^b axis at a distance $l > 0$ (bottom heavy) from the CB, i.e, $r_g = l i_3$ where i_3 denotes a unit vector (in body coordinates) along the e_3^b axis (see Fig. 2). Also let i_g denote a unit vector (in inertial coordinates) in the direction of gravity, i.e. along the e_3^r axis. Let m be the mass of the vehicle and J_b the inertia matrix for the vehicle. The moment applied to the body due to gravity, expressed in body coordinates is given by

$$r_g \times R^{\mathrm{T}} m g i_g = -mgl(\Gamma \times i_3) \quad \text{where } \Gamma = R^{\mathrm{T}} i_g.$$

The Lagrangian is the sum of the kinetic energy due to the body plus fluid and the potential $2mgl(i_g \cdot R i_3)$ that accounts for the moment contribution due to noncoincident center of mass and CB. The Lagrangian $L : TSE(3) \to \mathbb{R}$ is then given by

$$L(R, r, \dot{R}, \dot{r}) = \tfrac{1}{2}\left(\Omega^{\mathrm{T}} J \Omega + 2\Omega^{\mathrm{T}} D v + v^{\mathrm{T}} M v + 2mgl(i_g \cdot R i_3)\right),$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)^{\mathrm{T}}$ denotes the angular velocity and $v = (v_1, v_2, v_3)^{\mathrm{T}}$ denotes the components of the linear translational velocity along the body-fixed frame, m denotes the mass, $J = J_b + \Theta_{11}$, $D = ml\hat{r}_g$ and $M = mI + \Theta_{22}$ (I is the 3×3 identity matrix). Here Θ_{11} and Θ_{22} denote the added inertia and added mass terms, respectively (see Lamb, 1945). The Lagrangian is invariant under the action of the group

$$G = \{(R, r) \in SE(3) \mid R^{\mathrm{T}} i_g = i_g\} = SE(2) \times \mathbb{R}$$

and hence the Hamiltonian system on $T^*SE(3)$ (which is also left-invariant under the action of $SE(2) \times \mathbb{R}$) can be reduced to a Hamiltonian system on $\mathfrak{S}^*$, the dual of the Lie algebra of the semi-direct product $S = SE(3) \times_\rho \mathbb{R}^3$. The reduced Hamiltonian on $\mathfrak{S}^*$ is

$$\tilde{H}(\Pi, P, \Gamma) = \tfrac{1}{2}(\Pi^{\mathrm{T}} A \Pi + 2\Pi^{\mathrm{T}} B^{\mathrm{T}} P + P^{\mathrm{T}} C P - 2mgl(\Gamma \cdot i_3)),$$

where

$$A = (J - DM^{-1}D^{\mathrm{T}})^{-1}, \quad B = -CD^{\mathrm{T}} J^{-1},$$
$$C = (M - D^{\mathrm{T}} J^{-1} D)^{-1}, \quad \Pi = J\Omega + Dv,$$
$$P = Mv + D^{\mathrm{T}}\Omega, \quad \text{and} \quad \Gamma = R^{\mathrm{T}} i_g.$$

The Lie–Poisson tensor on $\mathfrak{S}^*$ is given by

$$\Lambda = \begin{bmatrix} \hat{\Pi} & \hat{P} & \hat{\Gamma} \\ \hat{P} & 0 & 0 \\ \hat{\Gamma} & 0 & 0 \end{bmatrix}.$$

In the rest of this paper the notation $\hat{}$ defines a map $\hat{}: \mathbb{R}^3 \to so(3)$, such that $\hat{\alpha}\beta = \alpha \times \beta, \alpha, \beta \in \mathbb{R}^3$. Thus

$$\hat{\alpha} = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}.$$

The Lie–Poisson reduced equations (Leonard, 1997 for a complete description of reduction procedure) are then given by

$$\dot{\mu}_i = \{\mu_i, \tilde{H}\}_-(\mu) + \Sigma_{i=1}^m \tilde{Y}_i(\mu) u_i, \quad \mu \in \mathfrak{S}^*,$$

where $\mu = (\Pi, P, \Gamma)^{\mathrm{T}}$ and Y_i's denote the control vector fields projected appropriately. In the rest of the paper, we assume that the underwater vehicle is underactuated and we have only three controls u_1, u_2, u_3 such that u_1 and u_2 provide pure torques and u_3 provides a pure force. Hence the reduced dynamics can be written explicitly as

$$\begin{aligned} \dot{\Pi} &= \Pi \times (A\Pi + B^{\mathrm{T}} P) + P \\ &\quad \times (CP + B\Pi) - mgl\Gamma \times i_3 + U_1, \\ \dot{P} &= P \times (A\Pi + B^{\mathrm{T}} P) + U_2, \\ \dot{\Gamma} &= \Gamma \times (A\Pi + B^{\mathrm{T}} P), \end{aligned} \tag{17}$$

where $U_1 = (u_1, u_2, 0)^{\mathrm{T}}$ and $U_2 = (u_3, 0, 0)^{\mathrm{T}}$. The reduced dynamics (17) along with the kinematics

$$\dot{r} = R\hat{\Omega}, \tag{18}$$

$$\dot{R} = Rv, \tag{19}$$

define the dynamics on $T^*SE(3)$.

In the case of coincident CG and CB (i.e. $l = 0$), $D = 0$. Hence the Lagrangian is given by

$$L = \tfrac{1}{2}(\Omega^{\mathrm{T}} T \Omega + v^{\mathrm{T}} M v).$$

The Hamiltonian system on $T^*SE(3)$ is left-invariant under the $SE(3)$ action of rotations and translations, and we can derive a set of reduced Lie–Poisson equations on $se(3)^*$. The Poisson tensor on $se(3)^*$ is given by

$$\Lambda(\mu) = \Lambda(\Pi, P) = \begin{bmatrix} \hat{\Pi} & \hat{P} \\ \hat{P} & 0 \end{bmatrix},$$

where $\Pi = J\Omega$ and $P = Mv$. (This was known to Novikov (see Novikov & Shmel'tser, 1984 and references therein) and other Russian researchers in the early 1980s). The Lie–Poisson reduced equations are given by

$$\dot{\Pi} = \Pi \times (A\Pi) + P \times CP + U_1, \tag{20}$$

$$\dot{P} = P \times A\Pi + U_2. \tag{21}$$

4. Controllability

In this section we present our main results on the controllability of the class of mechanical systems discussed in Section 2. We begin with a brief review of some familiar concepts in controllability of nonlinear systems (we refer the reader to Nijmeijer & van der Schaft, 1990, Chapter 3, for more details). Consider an affine nonlinear control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m. \tag{22}$$

Assume $x \in M \subset \mathbb{R}^n$, where M is a smooth manifold. Let $x(t, 0, x_0, u)$ denote the solution of (22) for $t \geqslant 0$ for a particular input function $u(\cdot)$ and initial condition $x(0)=x_0$. Let $R^V(x_0, T)=\{x \in M \mid$ there exists an admissible input u: $[0, T] \rightarrow U$ such that $x(t, 0, x_0, u) \in V$, $0 \leqslant t \leqslant T$ and $x(T) = x\}$ denote the reachable set from x_0. System (22) is *locally accessible* if given any $x_0 \in M$, $R^V(x_0, \leqslant T) = \bigcup_{\tau \leqslant T} R^V(x_0, \tau)$ contains a nonempty open set of M for all neighborhoods V of x_0 and all $T > 0$, and *locally strongly accessible* if given any $x_0 \in M$, then for any neighborhood V of x_0, $R^V(x_0, T)$ contains a nonempty open set for any $T > 0$ sufficiently small. System (22) is *controllable* if given any $x_0 \in M$, $\bigcup_{0 \leqslant T \leqslant \infty} R^V(x_0, \leqslant T) = M$, i.e. for any two points x_1, x_2 in M there exists a finite time T and an admissible function $u: [0, T] \rightarrow U$ such that $x(t, 0, x_1, u) = x_2$. Let $L(x) = \{\operatorname{span} X(x) \mid X$ vector field in $\mathscr{L}\}$, $x \in M$ where $\mathscr{L}$, the accessibility Lie algebra, is the smallest subalgebra of the Lie algebra of vector fields on M that contains $f, g_1, \ldots, g_m$. Let $L_0(x) = \operatorname{span}\{X(x) \mid X$ vector field in $\mathscr{L}_0\}, x \in M$ where $\mathscr{L}_0$ is the smallest Lie subalgebra which contains $g_1, \ldots, g_m$ and satisfies $[f, X] \in \mathscr{L}_0$, $\forall X \in \mathscr{L}_0$. Recall that in terms of the accessibility algebras, system (22) is *locally accessible* iff $\dim L(x) = n$, $\forall x \in M$, and *locally strongly accessible* if $\dim L_0(x) = n$, $\forall x \in M$. The condition that $L(x) = T_x M$, $\forall x \in M$ is often referred to as the Lie algebra rank condition (LARC).

If (22) satisfies the LARC, i.e. it is locally accessible, and in addition if $f = 0$, then we know from Chow's theorem (Hermann, 1968) that LARC implies that the system is controllable. While the kinematic equations of motion can often be written as a drift-free system, once dynamics are included LARC does not necessarily imply controllability. In this setting, proving controllability is usually much harder than proving accessibility. Our approach for the class of mechanical systems under discussion begins with an analysis of the reduced dynamics. We prove conditions under which the Lie–Poisson reduced dynamics are WPPS, and exploit WPPS combined with LARC to conclude controllability. Before we present our results we introduce some definitions and related theorems regarding Poisson stable systems. We follow the development in Dayawansa (1994), Lian et al. (1994) and Nemytskii and Stepanov (1960).

4.1. Poisson stability and controllability of reduced dynamics

Let X be a smooth complete vector field on M and let $\phi_t^X(\cdot)$ denote its flow. A point $p \in M$ is called *positively Poisson stable* for X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$, such that $\phi_t^X(p) \in V_p$. The vector field X is called positively Poisson stable if the set of Poisson stable points for X is dense in M. A point $p \in M$ is a nonwandering point of X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$ such that $\phi_t^X(V_p) \cap V_p \neq \phi$, where $\phi_t^X(V_p) = \{\phi_t^X(q) \mid q \in V_p\}$. One should observe here that though positive Poisson stability of a vector field is a sufficient condition that the nonwandering set is the entire manifold M, there could exist weaker conditions under which the nonwandering set is M. This gives rise to the definition: a vector field X is called WPPS if its nonwandering set is M. The following theorem on controllability is due to Lian et al. (1994). Earlier versions of this theorem where the hypothesis required f to be only Poisson stable, are due to Lobry (1974), Crouch, Pritchard, and Carmichael (1980).

Theorem 4.1 (Lian et al., 1994). *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m,$$

where U contains $\{u \mid |u_i| \leqslant M_i \neq 0,\ i, \ldots, m\}$ is such that f is a WPPS vector field, then the system is controllable if the accessibility LARC is satisfied.

A natural question that arises whether there is a sufficiently large class of vector fields that are WPPS. In the setting of Hamiltonian vector fields on bounded symplectic manifolds this question is answered by the Poincaré recurrence theorem stated below.

Theorem 4.2 (Poincaré recurrence theorem, Arnold, 1989; Nemytskii & Stepanov, 1960). *Let ψ be a volume-preserving continuous bijective map on a bounded region D onto itself. Then in any neighborhood U of any point in D, there exists a point $x \in U$ which returns to U after a repeated application of the mapping, i.e. $\psi^n(x) \in U$ for some finite integer n.*

Hamiltonian vector fields on symplectic manifolds are volume preserving. Hence, if in addition, the flows are restricted to a bounded set, or live on a bounded manifold, then it follows from Theorem 4.2 that a time-independent Hamiltonian vector field on a bounded symplectic manifold is WPPS.

As is easily observed from the dynamics of the jetpuck and underwater vehicles, the state space of these systems is not a bounded manifold and hence one cannot easily conclude the WPPS nature of the drift vector field in these cases.

In the setting of reduced dynamics where the drift vector field is Lie–Poisson we make the following observation.

Theorem 4.3. *Let G be a Lie group that acts on itself by left (right) translations. Let H: $T^*G \to \mathbb{R}$ be a left (right) invariant Hamiltonian. Then,*

(i) *If G is a compact group, the coadjoint orbits of $\mathfrak{G}^* = T^*G/G$ are bounded and the Lie–Poisson reduced Hamiltonian vector field $X_{\tilde{H}}$ is WPPS.*

(ii) *If G is a noncompact group then the Lie–Poisson reduced Hamiltonian vector field $X_{\tilde{H}}$ is WPPS if there exists a function $V : \mathfrak{G}^* \to \mathbb{R}$ such that $V(\mu)$ is bounded below, $V(\mu) \to \infty$ as $\|\mu\| \to \infty$ and $\dot{V} = 0$ along trajectories of the system.*

Here $\tilde{H}$ is the induced Hamiltonian on the quotient manifold $\mathfrak{G}^* = T^*G/G$ and $\{\cdot,\cdot\}_{-(+)}$ is the induced minus (plus) Lie–Poisson bracket on the quotient manifold $\mathfrak{G}^* = T^*G/G$.

Proof. (i) The projection $\lambda : T^*G \to \mathfrak{G}^*_-$ is a Poisson map, and the Poisson manifold $\mathfrak{G}^*_-$ is symplectically foliated by coadjoint orbits, i.e. it is a disjoint union of symplectic leaves that are just the coadjoint orbits. Any Hamiltonian system on $\mathfrak{G}^*_-$ leaves invariant the symplectic leaves and hence restricts to a canonical Hamiltonian system on a leaf. To study the dynamics of a particular system with initial condition $\mu(0) \in \mathfrak{G}^*_-$, we, therefore, restrict attention to the coadjoint orbit through $\mu(0)$. By hypothesis, each coadjoint orbit is compact. The flow starting at $\mu(0)$ preserves the symplectic volume measure on the orbit. Hence by the Poincaré Recurrence Theorem, we know that for almost every point $p \in \mathfrak{G}^*_-$ and any neighborhood V_p of p there exists a time $t > T$ such that $\phi_t^X(p)$ returns to V_p i.e. $X_{\tilde{H}}$ is WPPS.

(ii) Let $D = \{\mu \mid V(\mu) \leqslant E\}$, and let Orb$(\cdot)$ denote the coadjoint orbit through $\mu(0)$ in $\mathfrak{G}^*_-$. Then the integral curve of $X_{\tilde{H}}$ starting at $\mu(0) \in D$ lies entirely in the set $S = D \cap \mathrm{Orb}(\cdot)$. Since S closed and bounded in $\mathfrak{G}^*_-$, it is compact in Orb$(\cdot)$, and hence as before $X_{\tilde{H}}$ is WPPS. □

In many situations the function $H_\phi = \tilde{H} + \phi(C_i)$, where $\tilde{H}$ is the reduced Hamiltonian and C_i, a Casimir function, is a good choice for $V(\cdot)$. (Here a Casimir function is one whose Poisson bracket, with any function on $\mathfrak{G}^*$ is always zero).

Remark 4.4. In our present setting of Lie–Poisson reduced dynamics, WPPS conditions in Theorem 4.1 can be verified whenever the hypotheses of Theorem 4.3 hold. Once WPPS of the drift vector field has been established Theorem 4.1 can be used to conclude controllability.

Remark 4.5. As observed in Section 3, (in the example of the underwater vehicle) often the dynamics on T^*G are not invariant under the whole group G, but some subgroup of it. In such situations it might be possible to write down the reduced dynamics, using the semidirect product reduction theorem on the dual of the Lie algebra of a different group S which is a semidirect product. As these dynamics on $\mathfrak{S}^*$ are still Lie–Poisson, Theorem 4.3 still applies.

Applying the above results to the examples discussed in Section 3 we have the following results.

Proposition 4.6. *The reduced jetpuck dynamics defined by (15) are controllable if* $\sin\phi \neq 0$.

Proof. We first show that LARC is satisfied. To show that

$$\dim(\mathrm{span}\,\mathscr{L}_{\{f,g\}})(p) = 3, \quad \forall p \in se(2)^*,$$

where $f = (P_2\Pi/I, -P_1\Pi/I, 0)^{\mathrm{T}}$ and $g = (\alpha, \beta, d\beta)^{\mathrm{T}}$, observe that

$$\det(g, [[f,g],g], [[f,g],[[f,g],g]])$$

$$= \det \begin{bmatrix} \alpha & 2\frac{d}{I}\beta^2 & -2\frac{d^2}{I^2}\beta^2\alpha \\ \beta & -2\frac{d}{I}\beta\alpha & -2\frac{d^2}{I^2}\beta^3 \\ d\beta & 0 & 0 \end{bmatrix}$$

$$= -4\frac{(d\beta)^4}{I^3}(\beta^2 + \alpha^2)$$

$$= -4\frac{(d\beta)^4}{I^3} \quad (\text{since } \alpha^2 + \beta^2 = 1).$$

Hence $\dim(\mathrm{span}\,\mathscr{L}_{\{f,g\}})(p) = 3$, $\forall p \in se(2)^*$ as long as $\beta = \sin\phi \neq 0$, i.e. as long as the line of action of F (see Fig. 3) does not pass through the center of mass.

We observe that the reduced Hamiltonian

$$\tilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m} \tag{23}$$

is bounded below, radially unbounded and is such that $\dot{\tilde{H}} = 0$. Hence it follows from Theorem 4.3 that f is WPPS and hence from Theorem 4.1 we conclude that the jetpuck dynamics are controllable. In fact one observes that every orbit of f is periodic and hence f is trivially Poisson stable. □

Remark 4.7. Observe that the coadjoint orbits in $se(2)^*$ are cylinders

$$\{(P_1, P_2, \Pi) \in \mathbb{R}^3 \mid P_1^2 + P_2^2 = \text{constant } \neq 0\}.$$

The surfaces defined by $D = \{(P_1, P_2, \Pi) \mid P_1^2/2m + P_2^2/2m + \Pi^2/2I = \text{const}\}$ are ellipsoids. From Theorem 4.3 the integral curves of the vector field $(P_2\Pi/I)(\partial/\partial P_1) - (P_1\Pi/I)(\partial/\partial P_2)$ are restricted to a connected component of the set $S = D \cap \mathrm{Orb}(\cdot)$, which in this case is simply S^1 (see Fig. 3).

Observe that the jetpuck dynamics are similar to those of the reduced dynamics of a rigid body, symmetric about any one principal axis, with one control torque aligned with an axis having a nonzero axis component along all principal axes, allowing us to make similar controllability conclusions (cf., Baillieul, 1981; Crouch, 1984; Manikonda, 1998).

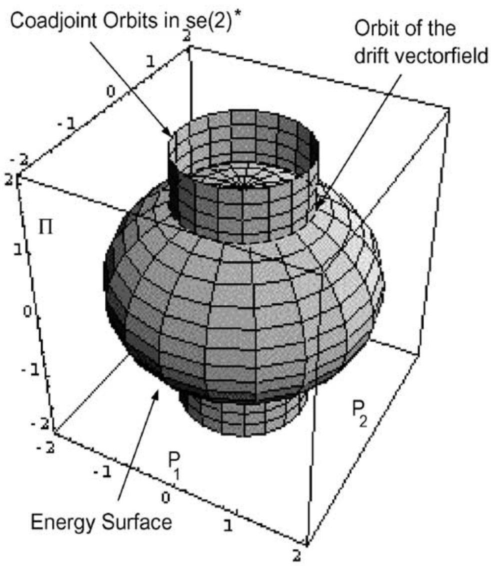

Fig. 3. Energy surface and coadjoint orbits in $se(2)^*$

In the setting of the autonomous underwater vehicle with coincident center of mass and CB, we assume that the vehicle is an ellipsoid with semiaxes l_1, l_2 and l_3 where l_i lies along the e_i^b axis. Assuming that the principal axes of the vehicle and the principal axes of the displaced fluid are the same we have

$$J = \mathrm{diag}(I_1, I_2, I_3) \quad \text{and} \quad M = \mathrm{diag}(m_1, m_2, m_3). \tag{24}$$

The reduced dynamics with these assumptions are

$$\dot{x} = f(x) + g_1 u_1 + g_2 u_2 + g_3 u_3, \tag{25}$$

where $x = (\Pi_1\ \Pi_2\ \Pi_3\ P_1\ P_2\ P_3)^{\mathrm{T}}$ and the vector fields f, g_1, g_2 and g_3 are given by

$$\begin{aligned} f = & \left(\frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2 - m_3}{m_2 m_3}P_2 P_3\right)\frac{\partial}{\partial \Pi_1} \\ & + \left(\frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3 - m_1}{m_3 m_1}P_3 P_1\right)\frac{\partial}{\partial \Pi_2} \\ & + \left(\frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1 - m_2}{m_1 m_2}P_1 P_2\right)\frac{\partial}{\partial \Pi_3} \\ & + \left(\frac{P_2\Pi_1}{I_3} - \frac{P_3\Pi_2}{I_2}\right)\frac{\partial}{\partial P_1} \\ & + \left(\frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3}\right)\frac{\partial}{\partial P_2} + \left(\frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1}\right)\frac{\partial}{\partial P_3}, \end{aligned}$$

$g_1 = \partial/\partial\Pi_1$, $g_2 = \partial/\partial\Pi_2$, and $g_3 = \partial/\partial P_1$, respectively.

Proposition 4.8. *The Lie–Poisson reduced dynamics, defined by* (25), *of the underwater vehicle with coincident CB and center of gravity are controllable if* $I_1 \neq I_2$.

Proof. Choose $V(\Pi, P) = \frac{1}{2}(\Pi^{\mathrm{T}} A \Pi + P^{\mathrm{T}} C P)$, where $A = J^{-1}$ and $C = M^{-1}$ are positive-definite symmetric matrices. Observing that V is radially unbounded and $\dot{V} = 0$ along trajectories of (25), we can conclude that f is WPPS. Further, we have

$$[[f, g_1], g_2] = \frac{(I_1 - I_2)}{I_1 I_2}\frac{\partial}{\partial \Pi_3}, [[f, g_2], g_3] = \frac{1}{I_2}\frac{\partial}{\partial P_3},$$

$$[[[f, g_2], [f, g_3]], g_1] = \frac{(I_1 - I_2)}{I_1 I_2 I_3}\frac{\partial}{\partial P_2}.$$

Treating vector fields f and g_i's as column vectors and observing that

$$\begin{aligned} & \det(g_1, g_2, g_3, [[f, g_1], g_2], [[f, g_2], g_3], [[[f, g_2], [f, g_3]], g_1]) \\ & \quad = \frac{(I_1 - I_2)^2}{I_1^2 I_3^3 I_3} \neq 0 \end{aligned} \tag{26}$$

if $I_1 \neq I_2$, i.e. $\dim(\mathrm{span}\, \mathscr{L}_{\{f, g_1, g_2, g_3\}})(p) = 6$, $\forall p \in se(3)^*$, and that f is WPPS controllability, the result follows from Theorem 4.3. □

Proposition 4.9. *The Lie–Poisson reduced Hamiltonian vector field corresponding to the reduced dynamics of the underwater vehicle with noncoincident center of mass and buoyancy* (17) *defined on* $\mathfrak{S}^*$ *is WPPS.*

Proof. Choose $V(\Pi, P, \Gamma) = \tilde{H}(\Pi, P, \Gamma) + \Gamma^{\mathrm{T}}\Gamma$. Observing that V is radially unbounded and that $\dot{V} = 0$ along trajectories of (17) the result follows from Theorem 4.3. □

4.2. Cotangent space controllability

In this section, we exploit the reduction procedure to prove controllability properties of the unreduced dynamics. We consider the cases when the symmetry group G is compact and noncompact, separately. If the symmetry group G is compact we show that the hypotheses of Theorem 4.3 are sufficient to conclude that the unreduced drift vector field X_H too is WPPS. To prove this we need the following lemma.

Lemma 4.10. *Let G be a compact Lie group whose action $\Phi: G \times M \to M$ on a manifold M is free. Let $\pi: M \to M/G$ denote the projection map. Then $D = \pi^{-1}(\tilde{D})$ is compact iff $\tilde{D} \subset M/G$ is compact, i.e the projection map π is a proper map.*

Proof. Assume that D is compact. Since G is compact Φ is proper. Hence π is a smooth submersion. Since π is a smooth submersion, if D is compact, then $\tilde{D} = \pi(D)$ is compact.

Now assume that $\tilde{D}$ is compact. Let $\{y_k\}$ be a sequence in $D = \pi^{-1}(\tilde{D})$. Let $\{x_k\} = \{\pi(y_k)\}$. $\{x_k\} \in \tilde{D}$, and since $\tilde{D}$ is compact $\{x_k\}$ has convergent subsequence $\{x_{k_j}\}$ that converges to $x^* \in \tilde{D}$. Now consider the subsequence $\{y_{k_j}\}$ such that $y_{k_j} \in \pi^{-1}(x_{k_j})$.

Since the action is free and proper, M is a principal fiber bundle over M/G (Abraham & Marsden, 1977), and we can find an open set $U \subset \tilde{D} \subset M/G$ such that $x^* \in U$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times G$. Let the diffeomorphism be given by $\phi: U \times G \to \pi^{-1}(U)$. Then $\pi(\phi(x,g)) = x$ for all $x \in U,\ g \in G$.

Since $x_{k_j} \to x^*$, there exists a N such that for all $j \geqslant N$, $x_{k_j} \in U$. Therefore, for all $j \geqslant N$, $y_{k_j} \in \pi^{-1}(U)$. Since ϕ is a diffeomorphism, it has a differentiable inverse ϕ^{-1}. This defines a sequence $g_j \in G$ by $\phi^{-1}(y_{k_j}) = (x_{k_j}, g_j)$. Since G is compact, there is a subsequence g_{j_p} that converges to a $g^* \in G$. Therefore, the sequence $\phi^{-1}(y_{k_{j_p}})$ converges to $(x^*, g^*) \in U \times G$. Since ϕ is a diffeomorphism, it follows that $y_{k_{j_p}}$ converges to $\phi(x^*, g^*) \in \pi^{-1}(U) \subset \pi^{-1}(\tilde{D})$. □

Theorem 4.11. *Let G be a compact Lie group whose action on a Poisson manifold M is free and proper. A G-invariant Hamiltonian vector field X_H defined on M is WPPS if there exists a function $V: M/G \to \mathbb{R}$ that is proper, bounded below and $\dot{V} = 0$ along trajectories of the projected vector field $X_{\tilde{H}}$ defined on M/G.*

Proof. Let $\tilde{D} = \{\mu \mid V(\mu) \leqslant E,\ \mu \in M/G\}$. Then the integral curve of $X_{\tilde{H}}$, denoted by $\phi_t^{X_{\tilde{H}}}(\mu_0)$ starting at $\mu_0 \in \tilde{D}$ lies entirely in $\tilde{D}$. Since $\tilde{D}$ is closed and bounded in M/G, it is compact. Let $\phi_t^{X_H}(x_0)$ be the integral curve of the Hamiltonian vector field X_H starting at x_0, at $t = 0$. At any given time $t' > 0$, $\phi_{t'}^{X_H}(x_0) \in \pi^{-1}(\phi_{t'}^{X_{\tilde{H}}}(\mu_0))$ where $\mu_0 = \pi(x_0)$. But $\phi_t^{X_{\tilde{H}}}(\mu_0)) \in \tilde{D},\ \forall t > 0$. Hence $\phi_{t'}^{X_H}(x_0) \in \pi^{-1}(\tilde{D})$. Since $\tilde{D}$ is compact from Lemma 4.10 $\pi^{-1}(\tilde{D})$ is compact and the integral curve of the Hamiltonian vector field X_H starting at x_0 is restricted to the compact set $\pi^{-1}(\tilde{D})$. To study the dynamics of X_H through x_0 we restrict ourselves to the symplectic leaf, induced by the Poisson bracket on M, passing through x_0. Let Σ be the symplectic leaf passing through x_0. (If the Poisson bracket on M is the Lie–Poisson bracket then Σ is the coadjoint orbit through x_0.). Any Hamiltonian system on M leaves invariant the symplectic leaves and hence restricts to a canonical Hamiltonian system on a leaf. Further Hamiltonian flows preserve the symplectic volume measure on the leaf. Since the integral curve $\phi_t^{X_H}(x_0)$ lies entirely in $W = \pi^{-1}(\tilde{D}) \cap \Sigma$, which is compact in Σ from Poincaré Recurrence Theorem, we know that for almost every point $p \in M$ and any neighborhood V_p of p there exists a time $t > T$ such that $\phi_t^X(p)$ returns to V_p, i.e. X_H is WPPS.

As mentioned earlier, having concluded the WPPS nature of the Hamiltonian vector field, if the Hamiltonian control system on M satisfies the LARC, then from Theorem 4.1 controllability can be concluded. The conclusion on the controllability of the unreduced dynamics where G is compact is similar in spirit to that of San Martin and Crouch (1984).

In the present setting of jetpuck and underwater vehicles we observe that though $SE(n)$, $n = 2, 3$ is not a compact group, it is a semidirect product, i.e. $G = H \times_\rho V$ where $H = SO(n)$ is compact and $V = \mathbb{R}^n$ is a vector space. For semidirect products one observes that $G/V \cong H$ (Rose, 1978). Hence reduction of G-invariant dynamics can be performed in two stages. First by V, to obtain dynamics on $H \times \mathfrak{G}^*$, and then by H to obtain the reduced dynamics on $\mathfrak{G}^*$. Hence under appropriate LARC assumptions we can conclude the reduced dynamics on $H \times \mathfrak{G}^*$ are controllable iff the Lie–Poisson reduced dynamics on T^*G/G are controllable.

Applying these results to the examples discussed earlier we have the following results.

Proposition 4.12. *The reduced dynamics of the jetpuck defined on $SO(2) \times se(2)^*$ are controllable if $\sin\phi \neq 0$.*

Proof. Let

$f = (\Pi/I, P_2\Pi/I, -P_1\Pi/I, 0)^{\mathrm{T}}$

and

$g = (0, \cos\phi, \sin\phi, d\sin\phi)^{\mathrm{T}}$.

Observe that

$$\det(g, [f,g], [[f,g],g], [[f,g][[f,g],g]]) = -\frac{4d^5 \sin^5\phi}{I^4}.$$

Hence LARC is satisfied iff $\sin\phi \neq 0$. The proof follows from Proposition 4.6 and Theorem 4.11. □

Proposition 4.13 (Manikonda, 1998). *The reduced dynamics* (19)–(21) *of the underwater vehicle with coincident center of mass and CB, defined on $SO(3) \times \mathfrak{se}(3)^*$ are controllable if $I_1 \neq I_2$.*

In the setting where the symmetry group is noncompact we show that, under additional assumptions of equilibrium controllability, reduced space controllability is sufficient to conclude controllability on T^*G. Before we prove this result we define equilibrium controllability, a concept introduced in Lewis and Murray (1996). Consider a mechanical system with a G-invariant Hamiltonian and G-invariant forces. Assume that the Hamiltonian is quadratic and projects to $\tilde{H} = \mu^{\mathrm{T}}\mathbb{I}^{-1}\mu$, $\mu \in \mathfrak{G}^*$, where $\mathbb{I}: \mathfrak{G} \to \mathfrak{G}^*$ is the inertia tensor. Then the dynamics on T^*G can be written in the form

$$\dot{g} = g\mathbb{I}^{-1}\mu, \tag{27}$$

$$\dot{\mu} = \Lambda(\mu)\nabla\tilde{H} + \sum_{i=1}^{m} f^i u_i. \tag{28}$$

Here $\Lambda(\mu)$ is the Lie–Poisson tensor defined on $\mathfrak{G}^*$. System (27)–(28) is *equilibrium controllable* if for any $(g_1, 0)$, $(g_2, 0)$ there exists a time $T > 0$ and an admissible input $u: [0,T] \to U$ such that the solution $(g(t), \mu(t))$ of (27)–(28) with initial conditions $(g(0), \mu(0)) = (g_1, 0)$ satisfies $(g(T), \mu(T)) = (g_2, 0)$.

Theorem 4.14. *The mechanical system given by* (27)–(28) *is controllable if*

(i) *the system is equilibrium controllable, and*
(ii) *the reduced dynamics* (28) *are controllable.*

Proof. We need to show that there exists a $T > 0$ and an admissible control $u: [0,T] \to U$ such that given any (g_1, μ_1) and (g_f, μ_f) the solution $(g(t), \mu(t))$ satisfies $(g(0), \mu(0))=$

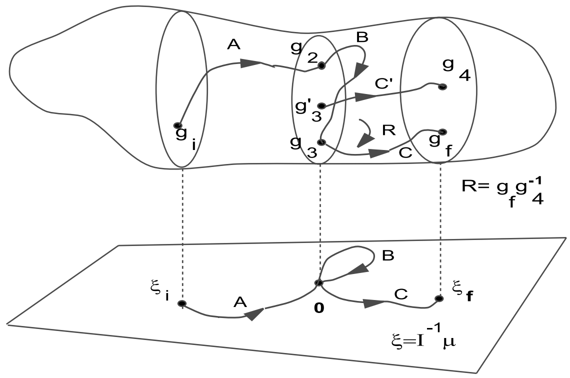

Fig. 4. Controllability on T^*G.

(g_1,μ_1) and $(g(T),\mu(T))=(g_f,\mu_f)$. Using the properties (i) and (ii) we construct such a control.

Assume that there exists a state $(g_3,0)$ and an admissible control u', such that u' will steer the system from $(g_3,0)$ to (g_f,μ_f) in finite time. (The existence of such a $(g_3,0)$ and u' is shown later.) The problem is now reduced to finding a control to steer the system from (g_1,μ_1) to $(g_3,0)$ which is done as follows.

Let $g(t,t_0,g_0,\mu(t))$ denote the solution of (27) at $t>t_0$ for a particular curve $\mu(t)\in\mathfrak{G}^*$ and initial condition g_0. Similarly, let $\mu(t,t_0,\mu_0,u(t))$ denote the solution of (28) at $t>t_0$ for a particular input u and initial condition μ_0, and let $\zeta(t,t_0,(g_0,\mu_0),u)$ denote the solution of (27)–(28) at $t>t_0$ for a particular input u and initial condition (g_0,μ_0).

1. Since the reduced dynamics are controllable there exists a $T_1>0$ and a control u_1 such that
2. $\zeta(T_1,0,(g_1,\mu_1),u_1)=(g_2,0)$ (for some g_2).
3. Since the dynamics are equilibrium controllable there exists a $T_2>T_1>0$ and a control u_2 such that $\zeta(T_2,T_1,(g_2,0),u_2)=(g_3,0)$.
4. Finally applying u' we have $\zeta(T_3,T_2,(g_3,0),u')=(g_f,x_f)$.

The existence of $(g_3,0)$ and u' is shown as follows. Find u_3 and T_3' such that $\mu(T_3',0,0,u_3)=\mu_f$. Existence of such a control follows from the reduced space controllability of (28). Apply the control u_3 to (27)–(28) with initial condition $\mu(0)=0$ and arbitrary $g(0)=g_3'$. Then $\zeta(T_3',0,(g_3',0),u_3)=(g_4,\mu_f)$ where g_4 need not be equal to g_f. Let $g(t,0,g_3',\mu(t))$ denote the solution to (27) where $\mu(t)=\mu(t,0,0,u_3)$. Let $R\in G$. Then by left invariance $\bar{g}=Rg(t,0,g_3',\mu(t))$ is a solution to (27)–(28). Choose R such that $\bar{g}(T_3')=Rg(T_3',0,g_3',\mu(t))=g_f$, i.e $R=g_fg_4^{-1}$ and hence $\bar{g}(t)=g_fg_4^{-1}g(t,0,g_3',\mu(t))$. Again from left invariance it implies that $\bar{g}(T_3',0,g_fg_4^{-1}g_3',\mu(t))=g_f$ or equivalently $\zeta(T_3',0,(g_fg_4^{-1}g_3',0),u_3)=(g_f,\mu_f)$. Hence choose $g_3=g_fg_4^{-1}g_3'$, $u'=u_3$ and $T_3=T_2+T_3'$. Fig. 4 depicts a pictorial representation of the proof. □

For a large class of systems given G-invariant dynamics, controllability of reduced space can be verified using Theorems 4.1 and 4.3. Equilibrium controllability can be verified using results in Bullo and Lewis (1996) and Lewis and Murray (1996), where the sufficient conditions for equilibrium controllability are presented. We briefly recall their results. Let $ad_\xi:\mathfrak{G}\to\mathfrak{G};\ \eta\mapsto[\xi,\eta]$ denote the adjoint map and ad_ξ^* denote its dual. Let

$$\dot{g}=g\xi, \tag{29}$$

$$\mathbb{I}\dot{\xi}=ad_\xi^*\mathbb{I}\xi+\sum_{i=1}^m f^iu_i \tag{30}$$

define G-invariant dynamics on TG.[2] Define the *symmetric product* $\langle\cdot:\cdot\rangle:\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}:\xi,\ \eta\mapsto\langle\xi:\eta\rangle$ as

$$\langle\xi:\eta\rangle=-\mathbb{I}^{-1}(ad_\xi^*\,\mathbb{I}\eta+ad_\eta^*\,\mathbb{I}\xi). \tag{31}$$

Let $B=\{b_1,\dots,b_m\}\subset\mathfrak{G}$ (a left invariant distribution on G) denote the input subspace. In the present setting $b_i=\mathbb{I}^{-1}f^i$. Let $\overline{Lie}_{\mathfrak{G}}(\mathscr{B})$ and $\overline{Sym}_{\mathfrak{G}}(\mathscr{B})$ denote the involutive and symmetric closure of $\mathscr{B}$ in $\mathfrak{G}$. A symmetric product is *bad* if it contains an even number of each of the vectors in $\mathscr{B}$. A symmetric product is *good* if it is not bad.

Theorem 4.15. [Bullo & Lewis, 1996; Lewis & Murray, 1996] *System* (29)–(30) *is equilibrium controllable if* $\operatorname{rank}(\overline{Lie}_{\mathfrak{G}}(\overline{Sym}_{\mathfrak{G}}(\mathscr{B})))=\dim(G)$ *and every bad symmetric product can be written as a linear combination of good symmetric products of lower degree.*

We now apply Theorem 4.14 to the autonomous underwater vehicle with coincident center of mass and CB. Controllability of reduced dynamics follows from Proposition 4.8. Equilibrium controllability of the dynamics are verified using Theorem 4.15.

Proposition 4.16. *The unreduced dynamics* (18)–(21) *of the autonomous underwater vehicle with coincident center of mass and CB, defined on* $T^*SE(3)$ (*or equivalently* $TSE(3)$) *are controllable if* $I_1\neq I_2$.

Proof. As shown in Theorem 4.14, controllability of (18)–(21) can be shown if controllability of reduced dynamics (20)–(21) and equilibrium controllability of (18)–(21) can be shown. In Proposition 4.8 controllability of reduced dynamics has already been shown. We now show that the dynamics are equilibrium controllable. Define J and M as in (24). Rewriting the reduced dynamics on $\mathfrak{se}(3)$ we have

$$\dot{\Omega}=J^{-1}(J\Omega\times\Omega+Mv\times v)+J^{-1}U_1, \tag{32}$$

$$\dot{v}=M^{-1}(Mv\times\Omega)+M^{-1}U_2, \tag{33}$$

[2] In Lewis and Murray (1996) it is assumed that the dynamics evolve on TG. Setting $\mu=\mathbb{I}\xi$, the two formulations (27)–(28) and (29)–(30) are equivalent.

where Ω and V are as defined in Section 3.3, $U_1=(u_1,u_2,0)$ and $U_2=(u_3,0,0)$. Thus the input space is spanned by the vectors

$$b_1=\begin{pmatrix}\frac{1}{I_1} & 0 & 0 & 0 & 0 & 0\end{pmatrix}^{\mathrm{T}},$$

$$b_2=\begin{pmatrix}0 & \frac{1}{I_2} & 0 & 0 & 0 & 0\end{pmatrix}^{\mathrm{T}},$$

and $b_3=\begin{pmatrix}0 & 0 & 0 & \frac{1}{m_1} & 0 & 0\end{pmatrix}^{\mathrm{T}}$.

Calculating the symmetric products we have

$$\langle b_1:b_2\rangle=\begin{pmatrix}0 & 0 & \frac{I_1-I_2}{I_1I_2I_3} & 0 & 0 & 0\end{pmatrix}^{\mathrm{T}},$$

$$\langle b_2:b_3\rangle=\begin{pmatrix}0 & 0 & 0 & 0 & -\frac{1}{I_2m_3} & 0\end{pmatrix}^{\mathrm{T}},$$

$$\langle b1:\langle b_2:b_3\rangle\rangle =\begin{pmatrix}0 & 0 & 0 & 0 & \frac{-1}{I_1I_2m_2} & 0\end{pmatrix}^{\mathrm{T}}.$$

Observe that

(i) $\mathscr{D}=\{b_1,b_2,b_3,\langle b_1:b_2\rangle,\langle b_2:b_3\rangle,\langle b1:\langle b_2:b_3\rangle\rangle\}$ spans $\mathbb{R}^6$ if $I_1\neq I_2$.

(ii) Every symmetric product in $\mathscr{D}$ is good, and from (i) every bracket of degree 4 or higher degree can be expressed a combination of lower-degree good brackets.

(iii) Every bad symmetric product of degree 2 is of the form $\langle b_i:b_i\rangle i=1,2,3$ and is equal to 0.

Hence it follows that the dynamics are equilibrium controllable and controllability of the unreduced system follows from Theorem 4.14. □

Remark 4.17. In the case of the jetpuck dynamics as we have only one input, every nontrivial second-order symmetric product is bad. Sufficient conditions for equilibrium controllability are not satisfied and hence we cannot conclude controllability of unreduced dynamics. In fact one observes that the jetpuck dynamics are not even small-time locally controllable (STLC) (see following section). As shown in the proposition below, one can only conclude strong accessibility (Nijmeijer & van der Schaft, 1990). □

Proposition 4.18. *The jetpuck dynamics defined on* $T^*SE(3)$ *are locally strongly accessible if* $\sin\phi\neq 0$.

Proof. Explicitly writing dynamics (15)–(16) observe that the drift vector field f is given by

$$f=\left(\frac{P_1\cos\theta}{m}-\frac{P_2\sin\theta}{m}\right)\frac{\partial}{\partial x} +\left(\frac{P_2\cos\theta}{m}+\frac{P_1\sin\theta}{m}\right)\frac{\partial}{\partial y}+\frac{\Pi}{I}\frac{\partial}{\partial\theta}$$

and the input vector field g is given by

$$g=\cos\phi\,\frac{\partial}{\partial P_1}+\sin\phi\,\frac{\partial}{\partial P_2}+d\sin\phi\,\frac{\partial}{\partial\Pi}.$$

We calculate the following brackets

$$\xi_1=[f,g],\quad \xi_2=[[f,g],g],\quad \xi_3=[f,[[f,g],g]],$$
$$\xi_4=[[f,g],[[f,g],g]],\quad \xi_5=[f,[[f,g],[[f,g],g]]].$$

Again treating g and ξ_i's as column vectors, observe that

$$\det[g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5]=-\frac{16d^8\sin^8\phi}{I^7\,m^2}.$$

Hence again if $\sin\phi\neq 0$, $\dim(\operatorname{span}\mathscr{L}_{\{f,g\}})(p)=6\ \forall p\in T^*SE(2)$. Also $[f,X]\in\operatorname{span}(g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\ \forall X\in\{g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5\}$. Hence the complete system is locally strongly accessible. □

4.3. Small-time local controllability

While showing controllability in systems can be quite difficult, one can often show that the system is STLC, as STLC can be determined by the "type" of brackets that span the Lie algebra. For the sake of completeness, in this section, we discuss the STLC properties of the mechanical systems discussed in Section 3.

Recall that control system (22) is said to be STLC (Sussmann, 1983, 1987) from $x_0\in M$ if it is locally accessible from x_0, and x_0 is in the interior of $R^V(x_0,\leqslant T)$ for all $T>0$ and each neighborhood V of x_0. If this holds for any $x_0\in M$ then the system is called STLC.

Let $X=\{X_0,\dots X_m\}$. Let $Br(X)$ denote the set of all possible "brackets" of elements of X. Let $\delta_i(B)$ denote the number of occurrences of X_i in $B\in Br(X)$. An element $B\in Br(X)$ is said to be *bad* if $\delta_0(B)$ is odd and $\delta_i(B)$ is even for each $i=1,\dots,m$. A bracket is *good* if it is not bad. Let S_m denote the permutation group on m symbols. For $\pi\in S_m$ and $B\in Br(X)$, define $\bar{\pi}(B)$ to be the bracket obtained by fixing X_0 and sending X_i to $X_{\pi(i)}$ for $a=1,\dots,m$. Now define

$$\beta(B)=\sum_{\pi\in S_m}\bar{\pi}(B).$$

Consider the bijection $\psi:X\to\{f,g_1,\dots,g_m\}$ which sends X_0 to f and X_i to g_i for $i=1,\dots m$, and define the evaluation map

$$Ev(\psi):\ \mathscr{L}(X)\to\mathscr{L}(\mathscr{F}):\sum_I\alpha_iX_i\mapsto\sum_I\alpha_i\psi(X_i)\alpha_i\in\mathbb{R}.$$

In Sussmann (1987) the following sufficient condition for STLC in terms of the Lie brackets and Lie algebra generated by the vector fields $\{f,g_1,\dots,g_m\}$ was given.

Theorem 4.19 (Sussmann, 1987). *Consider the bijection* $\psi:X\to\{f,g_1,\dots,g_m\}$ *which sends* X_0 *to* f *and* X_i *to* g_i *for* $i=1,\dots,m$. *Suppose that system* (22) *is such that every* *bad bracket $B \in Br(X)$ has the property that*

$$Ev_x(\psi)(\beta(B)) = \sum_{i=1}^{m} \alpha^i Ev_x(\psi)(C_i),$$

where C_i are good brackets in $Br(X)$ of lower degree than B and $\alpha_i \in \mathbb{R}$, $i = 1,\ldots,m$. Also suppose that (22) satisfies the LARC at x. Then (22) is STLC from x.

Hence if all bad brackets can be "neutralized" or can be expressed as a linear combination of good brackets of a lower degree then the system is STLC. In Sussmann (1983) the following necessary condition for single input systems was shown.

Theorem 4.20 (Sussmann, 1983). *Consider an analytic system*

$$\dot{x} = f_0(x) + f_1(x)u, \quad |u(t)| \leqslant A \tag{34}$$

and a point x_0 such that

$$[f_1,[f_0,f_1]](x_o) \notin \mathscr{S}^1(f_0 + \tilde{u}f_1, f_1)(x_0),$$

where $\mathscr{S}^1(X_1,X_2)$ is the linear span of X_1,X_2, and the brackets $(ad\,X_1)^j X_2$ for $j \geqslant 1$, and $\tilde{u}$ is such that $f_0(x_0) + \tilde{u}f_1(x_0) = 0$. Then (34) is not STLC from x_0.

Using this result one can conclude that the jetpuck dynamics are not STLC.

Proposition 4.21. *The unreduced jetpuck dynamics defined (15)–(16) on $T^*SE(2)$ are not STLC from the origin.*

Proof. It is sufficient to consider STLC of the reduced dynamics (15). With $\tilde{u} = 0$ observe that $\mathscr{S}^1(f,g)(0)$ is a one-dimensional space spanned by $\alpha(\partial/\partial P_1) + \beta(\partial/\partial P_2) + d\beta(\partial/\partial \Pi)$ while $[g,[f,g]](0) = -2(d/I)\beta^2(\partial/\partial P_1) + 2(d/I)\beta\alpha(\partial/\partial P_2)$. Hence $[g,[f,g]](0) \notin \mathscr{S}^1(f,g)(0)$. □

We now study the STLC of the underwater vehicle with coincident center of mass and CB, again assuming that the principal axes of the vehicle and the principal axes of the displaced fluid are the same.

Proposition 4.22. *The reduced underwater vehicle dynamics defined by (25) are STLC if $I_1 \neq I_2$.*

Proof. In Proposition 4.8 we already showed that the LARC was satisfied. Hence, we need to verify that all bad brackets can be expressed as a linear combination of good brackets of lower degree. One first observes that from Theorem 4.19 all bad brackets are of odd degree. From (26) it follows that all brackets of degree 6 or higher can be expressed a linear combination of lower order brackets. Further all brackets in (26) are good. Hence we need to only check for brackets of order 1, 3, and 5. The degree 1 bracket is f which is equal to 0 at the equilibrium $(\Pi, P) = (0,0)$. The degree 3 brackets are $[[f,g_i],g_i]$, $i = 1,2,3$ which are equal to 0 for all (Π, P). The degree 5 brackets can be broken into three sets: (i) $[[[[f,g_i],g_i],g_i],g_i]$, $i = 1,2,3$ which are again equal to zero since $[[f,g_i],g_i]=0$, $i=1,2,3$, (ii) $[[[f,g_i],f],[f,g_i]]$, $i=1,2,3$ and (iii) $[[[f,g_i],[f,[f,g_i]]]$, $i=1,2,3$. The brackets (ii) and (iii) are equal to zero at $(\Pi,P) = (0,0)$. Hence we conclude that the reduced dynamics are STLC. □

The above conclusions regarding STLC of the jetpuck and underwater vehicle dynamics can also be made using the formalism of equilibrium controllability, as discussed in Lewis and Murray (1996) and Bullo et al. (2000).

5. Conclusions

In this paper we presented results related to controllability of a class of nonlinear left-invariant mechanical systems with symmetry. Our results on controllability provide a manageable tool to check for controllability of a wide class of mechanical systems including the jetpuck and underwater vehicles. Future research includes the design of feedback laws to stabilize the origin of the reduced system, the use of periodic controls to generate loops in the base space, and thereby steer in the fiber, and the design of feedback laws based on center manifold techniques to stabilize relative equilibria (Krishnaprasad & Manikonda, 1998).

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