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CONTROL OF SWITCHED ELECTRICAL NETWORKS USING AVERAGING ON LIE GROUPS *

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Abstract

In this paper we apply the theory of averaging and motion control on Lie groups [1] to the problem of controlling energy transfers between dynamic storage elements in switched electrical networks. The switched networks of interest have bilinear state-space models in which the control u, representing the position of the switch, takes value in the set $\{0,1\}$. The corresponding state transition matrix can be described by a right-invariant system evolving on a matrix Lie group [2], and as such we can use our theory to derive high-order average approximations to the evolution of the state transition matrix. We show how to use these average solutions to control energy transfers for a simple network that models the conversion portion of a dc-dc converter. Our approach provides an alternative to the feedback approach of Sira-Ramirez [3] which is based on variable structure systems with sliding regimes. Our methodology is based on open-loop control (combined with feedback if desired) and thus ensures that prescribed energy transfers can be accomplished with a finite number of switchings. This avoids chattering problems sometimes associated with sustaining sliding motions.

1. Introduction

A class of nonlinear mechanical systems can be modelled as drift-free, left-invariant systems on Lie groups [1]:

$$\dot{X} = XU, \quad U(t) = \sum_{i=1}^{m} u_i(t) A_i. \tag{1}$$

$X(t)$ is a curve in the n-dimensional matrix Lie group G, $U(t)$ is a curve in the associated Lie algebra $\mathcal{G}$, $\{A_1, \ldots, A_n\}$ is a basis for $\mathcal{G}$, and $u_i(t)$ are control inputs. For example, the attitude of a spacecraft with zero angular momentum and internal rotors for control can be described by a curve $X(t)$ in the group of rotations $G = SO(3)$ which satisfies (1). In this case $\mathcal{G} = so(3) = \{B \in \Re^{3\times 3} \mid B + B^T = 0\}$, and $u_i(t)$, $i = 1, \ldots, m$, $m \leq 3$ are the controlled angular velocities of the rotors.

Averaging theory for system (1) was developed to construct small (ϵ) amplitude, periodic, open-loop controls to achieve prescribed motion for such nonlinear mechanical systems. The averaging theory allows one to formulate a high-order average approximation $\bar{X}(t) \in G$ to $X(t)$. The formula for $\bar{X}(t)$ admits a geometric interpretation which makes it possible to derive open-loop control laws that take $\bar{X}(t)$ exactly, and thus $X(t)$ approximately, from any $X_i \in G$ to any $X_f \in G$.

As Brockett has argued [4], the study of systems like (1) can be further motivated by problems associated with bilinear systems on $\Re^n$. This is because the evolution of the state transition matrix for these bilinear systems can be described by a right-invariant (though not necessarily drift-free) system on a matrix Lie group. In particular, Brockett and Wood [2, 5] have shown that switched electrical networks such as those used in power conversion applications can be modelled as bilinear systems with state transition matrices that evolve on matrix Lie groups such as $SO(k)$ and $SE(k)$ where

$$SO(k) \triangleq \{A \in \Re^{k\times k} | A^T A = I, \ \det(A) = 1\},$$

$$SE(k) \triangleq \{\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \in \Re^{(k+1)\times(k+1)} | A \in SO(k), b \in \Re^k\}.$$

In this paper we apply our averaging theory for system (1) to one type of problem in the control of switched electrical networks. Here, switching is performed in a periodic way with small (ϵ) period (which is equivalent to small (ϵ) amplitude in scaled time). The control problem is that of transfering energy between dynamic storage elements in a lossless network according to a prescribed path. In the Lie group framework the problem becomes one of determining switchings that drive the state transition matrix as desired.

Switched electrical networks play an important role in a variety of systems including dc-dc switchmode power converters which are of great value in many growing application areas such as in communication and data handling systems, portable battery-operated equipment, and uninterruptable power sources (see [6]). In standard practice, switching strategies (pulse-width-modulated controls) are derived based on state-space averaging methods with $O(\epsilon)$ accuracy. Increased accuracy is achieved with decreased ϵ which corresponds to a higher switching frequency. In our approach we consider higher-order (i.e., $O(\epsilon^p)$, $p > 1$) averaging so that we can increase accuracy by increasing p rather than decreasing ϵ. Brockett and Wood [5] derive high-order approximations of the state-space equations using a truncation of the Campbell-Baker-Hausdorff formula for the logarithm of the product of exponentials. Our work provides a justification via averaging theory on Lie groups for their formula.

Sira-Ramirez [3] has developed a feedback controller design for switched electrical networks based on variable structure systems theory and sliding regimes. Our controller design focuses primarily on open-loop control, although feedback can be added for robustness. One advantage of open-loop control is that energy transfers in

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the networks can be accomplished in a predictable, finite number of switchings. This eliminates the possibility of chattering which is sometimes associated with sustaining motion on a sliding regime.

In Section 2 we define the general problem, outline our controller design approach and describe our network example. Averaging theory is applied in Section 3 and controllability is determined in Section 4. In Section 5 we describe our controller design for the example. In Section 6 we give conclusions.

2. Problem Description and Approach

The (idealized) state-space bilinear equations for switched electrical networks generally take the form [2, 5]:

$$\dot{x} = (A_0+A_1u_1+\ldots+A_mu_m)x+(b_1u_1+\ldots+b_mu_m). \quad (2)$$

The state $x \in \Re^n$ represents inductor currents and capacitor voltages. Each control $u_i(t) \in \{0,1\}$ represents the position of a switch at time t. Typically, the network will have only one switch or a set of switches that change position synchronously such that control action can be represented by a single switch position u, also taking value in the set $\{0,1\}$. The vectors $b_1, \ldots, b_m$ typically represent constant power sources. We can always write (2) as a homogeneous equation by augmenting the state vector to be $(x \ \ 1)^T$. Thus, the state-space equations of interest become

$$\dot{x} = (A+Bu)x = (A(1-u)+(A+B)u)x. \quad (3)$$

We note that this is a bilinear system which always has a drift term, i.e., nontrivial dynamics when $u=0$.

From linear systems theory it is well known that the state transition matrix $\Phi(t) \in \Re^{n\times n}$ which describes the evolution of the state according to

$$x(t) \doteq \Phi(t)x(0) \quad (4)$$

satisfies the same equation (3) as x, i.e.,

$$\dot{\Phi} = (A+Bu)\Phi, \quad \Phi(0) = I. \quad (5)$$

It is also well known [4] that $\Phi(t)$ will evolve on the matrix Lie group G associated with the Lie algebra generated by A and B. Equation (5) describes a right-invariant system with drift on the Lie group G. If we define $X = \Phi^{-1} \in G$, then X satisfies

$$\dot{X} = X(-A-Bu). \quad (6)$$

Equation (6) describes a left-invariant system with drift on the Lie group G. Defining $A_1 = A$, $A_2 = A+B$, $u_1 = -(1-u)$ and $u_2 = -u$, we can rewrite (6) as

$$\dot{X} = X(A_1u_1 + A_2u_2). \quad (7)$$

System (7) is of the same form as (1). We have disguised the fact that the system has a drift term by defining two controls which, in fact, are not independent. However, by transforming our system into the form (7), we can apply averaging theory on Lie groups to determine high-order average approximations to the behavior of X and thus Φ.

The goal of the problem that we address is to specify switching controls that will drive the state of the system (3) from some initial condition x_i along a desired path to a final condition x_f. To solve the problem we choose k target points $x_1, \ldots, x_k$ between x_i and x_f along the desired path. We then focus on the problem of specifying an open loop switching control strategy to drive the state between successive target points. Feedback could be introduced for robustness in between steps, if desired.

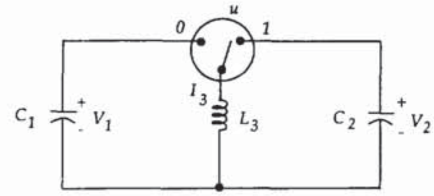

Figure 1: Network Example

From equation (4), the problem of controlling the evolution of $x(t)$ can be considered as a problem of controlling the evolution of $\Phi(t)$ (or alternatively $X(t)$). We compute the average approximation $\bar{\Phi}(t)$ of the solution to (5) and then determine the switching control that will drive $\bar{\Phi}(t)$ as desired exactly in order to drive $\Phi(t)$ as desired approximately.

The network example that we study in this paper is shown in Figure 1. This simple network can be considered as a model of the conversion portion of a dc-dc voltage converter [2]. The control problem is to convert energy from one capacitor to the other via the inductor. For example, suppose that at $t=0$, $V_1(0) = V_{10}$, $I_3(0) = 0$ and $V_2(0) = 0$. By appropriate switching, the energy in C_1 can be transfered to C_2 so that at some later time t', $V_1(t') = 0$, $I_3(t') = 0$ and $V_2(t') = V_2{}'$. The ratio $V_2{}'/V_{10}$ will depend on the ratio of the capacitances C_1/C_2 so that voltage scaling up or down can be achieved with an appropriate choice of capacitors. Of particular interest is controlling the energy transfers in this network to meet certain performance criteria. In Section 5 we design controls to transfer energy between capacitors while maintaining a constant nonzero current I_3 through the inductor.

The equations for this network are

$$\begin{aligned} C_1\dot{V}_1 &= (1-u)I_3 \\ C_2\dot{V}_2 &= uI_3 \\ L_3\dot{I}_3 &= -(1-u)V_1 - uV_2. \end{aligned} \quad (8)$$

Define $\hat{}: \Re^3 \to so(3)$ where $y = (y_1, y_2, y_3)^T$, by

$$\hat{y} = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}.$$

Let $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$. Define $A_i = \hat{e}_i$, $i = 1,2,3$. Then $\{A_1, A_2, A_3\}$ is a basis for $so(3)$. Now define the state vector of the network $x = (x_1, x_2, x_3)^T$ by $x_1 = \sqrt{C_1}V_1$, $x_2 = \sqrt{C_2}V_2$, $x_3 = \sqrt{L_3}I_3$, and let $\omega_1 = 1/\sqrt{C_1L_3}$ and $\omega_2 = 1/\sqrt{C_2L_3}$. Then (8) can be rewritten as

$$\dot{x} = (\omega_1A_2 - (\omega_1A_2+\omega_2A_1)u)x = (-u_1A_1 - u_2A_2)x \quad (9)$$

where $u_1 = \omega_2u$ and $u_2 = -\omega_1(1-u)$. Since the Lie bracket $[A_1, A_2] = A_1A_2 - A_2A_1 = A_3$, the Lie algebra generated by A_1 and A_2 is $so(3)$. Thus, the state transition matrix $\Phi(t)$ evolves on the Lie group $SO(3)$. Since an orthogonal matrix acting on a vector in $\Re^3$ preserves length, $x(t)$ evolves on a sphere in $\Re^3$. In particular, this confirms that energy, $\frac{1}{2}x^Tx$, is conserved.

The network of Figure 1 is representative of a class of interesting networks. With an increased number of capacitors and/or inductors in the network, the state will evolve on a higher dimensional sphere and the transition matrix will evolve on a higher dimensional orthogonal group $SO(k)$. If power sources and/or loads are introduced into the network then the state will no longer evolve on a sphere since energy may be added to or dissipated from the network. In some of these cases it can be shown that the state transition matrix evolves on the Euclidean group $SE(k)$ [2].

3. Averaging

In our previous work on averaging theory [1], we consider system (1) with small amplitude periodic controls. That is, we rewrite (1) as

$$\dot{X} = \epsilon X U, \quad U(t) = \sum_{i=1}^{m} u_i(t) A_i \tag{10}$$

where ϵ is a small parameter representing the amplitude of the control and $u(t) = (u_1(t), \ldots, u_m(t))$ is periodic in t with period $T = O(1)$. Given a metric d on the Lie group G, a pth-order average approximation $X^{(p)}$ satisfies $d(X(t), X^{(p)}(t)) = O(\epsilon^p)$, $\forall t \in [0, b/\epsilon]$, $b > 0$. In [1] we derive formulas for $X^{(p)}$, p a positive integer. The second-order average $X^{(2)}$ will be deemed sufficient for the present study although we could certainly use higher-order averages to improve the accuracy of our results.

To begin we make a few definitions. First we extend u to be a vector in $\Re^n$ by adding zeros, i.e., $u_{m+1} = \ldots = u_n = 0$. Then we let $u_{av} = (u_{av1}, \ldots, u_{avn})^T$, $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T$, U_{av} and $\tilde{U}$ be defined by

$$u_{avi} = \frac{1}{T}\int_0^T u_i(\tau)d\tau, \quad \tilde{u}_i(t) = \int_0^t u_i(\tau)d\tau, \tag{11}$$

$$U_{av} = \sum_{i=1}^{n} u_{avi} A_i, \quad \tilde{U}(t) = \sum_{i=1}^{n} \tilde{u}_i(t) A_i.$$

Next we define

$$Area_{ij}(T) = \frac{1}{2}\int_0^T (\tilde{u}_i(\sigma)u_j(\sigma) - \tilde{u}_j(\sigma)u_i(\sigma))d\sigma, \tag{12}$$

$$a_{ij}(t) = \frac{1}{2}\int_0^t (\tilde{u}_i(\sigma)u_j(\sigma) - \tilde{u}_j(\sigma)u_i(\sigma))d\sigma. \tag{13}$$

$Area_{ij}(T)$ can be interpreted as the area bounded by the curve described by $\tilde{u}_i$ and $\tilde{u}_j$ and the straight line described by $u_{avi}t$ and $u_{avj}t$ over one period. The *structure constants* Γ_{ij}^k associated to a given basis for the Lie algebra $\mathcal{G}$ are defined by

$$[A_i, A_j] = \sum_{k=1}^{n} \Gamma_{ij}^k A_k, \quad i, j = 1, \ldots, n$$

where $[\cdot,\cdot]$ is the Lie bracket on $\mathcal{G}$ defined by $[A, B] = AB - BA$.

To derive our averaging formulas we make use of a recursive solution to (10). In particular, we consider the single exponential representation [7] where for $X(0) = I$,

$$X(t) = e^{Z(t)} = \exp\{Z(t)\} \tag{14}$$

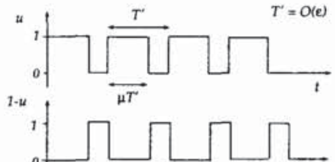

Figure 2: Switching Control

and $Z(t) \in \mathcal{G}$ is given by the infinite series:

$$Z(t) = \epsilon\int_0^t U(\tau)d\tau + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau + \ldots \tag{15}$$

Convergence conditions for this series are discussed in [1].

In accordance with our use of this recursive solution to (10), we define a metric on the Lie group G by transferring a metric on the Lie algebra $\mathcal{G}$ (a vector space) using a diffeomorphism. Let $\hat{S}$ be the largest neighborhood of $0 \in \mathcal{G}$ such that $\hat{\Psi} = \exp|_{\hat{S}} : \hat{S} \to G$ is one-to-one. Let $\hat{Q} = \hat{\Psi}(\hat{S}) \subset G$. Then $\hat{\Psi} : \hat{S} \to \hat{Q}$ is a diffeomorphism and we can define a metric $\hat{d} : \hat{Q} \times \hat{Q} \to \Re_+$ by

$$\hat{d}(X, Y) = d(\hat{\Psi}^{-1}(X), \hat{\Psi}^{-1}(Y))$$

where $d : \Re^n \times \Re^n \to \Re_+$ is given by $d(\alpha, \beta) = \|\alpha - \beta\|$. For simply connected Lie groups or Lie groups that have a connected center that is simply connected (such as $SO(3)$), Lazard and Tits [8] have shown that $\hat{S}$ and thus $\hat{Q}$ is relatively large. For a detailed discussion see [1].

The following theorem gives the formulas for $X^{(1)}$ and $X^{(2)}$. The proof for these results can be found in [1].

Theorem 1. Consider system (10). Let $b > 0$ be such that $\int_0^t \|U(\tau)\|d\tau < \delta$, $\forall t \in [0, b]$, where δ is small enough for convergence of (15). Assume that $U(t+T) = U(t)$, $\forall t > 0$. Let $X(0) = X_0 \in \hat{Q} \subset G$ and $Z_0 = \hat{\Psi}^{-1}(X_0) = O(\epsilon)$. Define for $p = 1, 2$,

$$X^{(p)}(t) = e^{Z^{(p)}(t)}, \quad Z^{(p)}(t) = \sum_{k=1}^{n} z_k^{(p)}(t) A_k + Z_0^{(p)} \tag{16}$$

$$z_k^{(1)}(t) = \epsilon u_{avk} t,$$

$$z_k^{(2)}(t) = \epsilon \tilde{u}_k(t) + \epsilon^2 \sum_{i,j=1; i<j}^{m} a_{ij}(t)\Gamma_{ij}^k.$$

If $\|Z_0 - Z_0^{(p)}\| = O(\epsilon^p)$ and $Z^{(p)}(t) \in \hat{S}$, $\forall t \in [0, b/\epsilon]$,

$$\hat{d}(X(t), X^{(p)}(t)) = O(\epsilon^p), \quad \forall t \in [0, b/\epsilon], \quad p = 1, 2.$$

Further, for $t = qT$, q an integer,

$$z_k^{(2)}(qT) = \epsilon qT u_{avk} + \epsilon^2 q \sum_{i,j=1; i<j}^{m} Area_{ij}(T)\Gamma_{ij}^k.$$

The relevant controls for switched electrical networks are illustrated in Figure 2. These controls have $O(1)$ amplitude and $O(\epsilon)$ period whereas the controls of system (10) studied in Theorem 1 have $O(\epsilon)$ amplitude and $O(1)$ period. The following corollary is a time-scaled version of Theorem 1 which provides $O(\epsilon)$ and $O(\epsilon^2)$ average

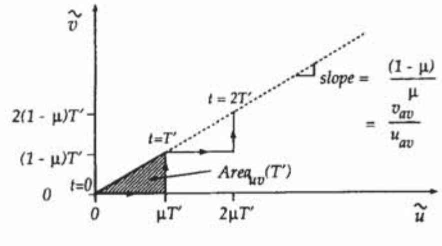

Figure 3: Geometric Interpretation of $Area_{uv}(T')$

approximations of the solution to (1) with t-periodic controls having $O(\epsilon)$ period. The proof, which applies time-scaling to the system considered in Theorem 1, can be found in detail in [1].

Corollary 1. Consider system (1). Assume that b is as in Theorem 1 and $U(t+T') = U(t)$, $\forall t > 0$, where $T' = O(\epsilon)$. Let $X(0) = X_0 \in \hat{Q} \subset G$ and $Z_0 = \hat{\Psi}^{-1}(X_0) = O(\epsilon)$. Define $X^{(p)}$, $Z^{(p)}$ by (16) and

$$\begin{aligned} z_k^{(1)}(t) &= u_{avk}t, \\ z_k^{(2)}(t) &= \tilde{u}_k(t) + \sum_{i,j=1;i<j}^{m} a_{ij}(t)\Gamma_{ij}^k. \end{aligned}$$

If $\|Z_0 - Z_0^{(p)}\| = O(\epsilon^p)$ and $Z^{(p)}(t) \in \hat{S}$, $\forall t \in [0,b]$,

$$\hat{d}(X(t), X^{(p)}(t)) = O(\epsilon^p), \quad \forall t \in [0,b], \ \ p = 1,2.$$

Further, for $t = qT'$, q an integer,

$$z_k^{(2)}(qT') = qT'u_{avk} + q \sum_{i,j=1;i<j}^{m} Area_{ij}(T')\Gamma_{ij}^k.$$

It is straightforward to compute the relevant terms in the average formulas of Corollary 1 for our controls in Figure 2. The variable μ, where $0 \le \mu \le 1$, referred to as the *duty ratio*, is defined as the fraction of the period T' for which $u = 1$. If we only concern ourselves with moments of time t when $t = qT'$, q an integer, then it is irrelevant when during a period $u = 1$, i.e., shifts in time have no effect on our results. Letting $v = 1 - u$ we find using definitions (11) and (12) that

$$u_{av} = \mu, \quad v_{av} = 1 - \mu, \quad Area_{uv}(T') = \frac{1}{2}T'^2\mu(1-\mu). \tag{17}$$

Figure 3 illustrates the geometric interpretation of the area term $Area_{uv}(T')$.

For the example network of Figure 1, we have by (9) that

$$\begin{aligned} \dot{\Phi} &= (-u_1A_1 - u_2A_2)\Phi, \quad \Phi(0) = I \\ \dot{X} &= X(u_1A_1 + u_2A_2), \quad X(0) = I \end{aligned} \tag{18}$$

where $A_1 = \hat{e}_1$, $A_2 = \hat{e}_2$, $u_1 = \omega_2 u$ and $u_2 = -\omega_1(1-u)$. Thus, from (17)

$$\begin{aligned} u_{av1} &= \omega_2\mu, \quad u_{av2} = -\omega_1(1-\mu), \quad (19) \\ Area_{12}(T') &= -\frac{1}{2}\omega_1\omega_2 T'^2\mu(1-\mu). \end{aligned}$$

Therefore, from Corollary 1, since the relevant structure constant is $\Gamma_{12}^3 = 1$ ($[A_1, A_2] = A_3$) the average state transition matrix solutions are for $p = 1,2$:

$$\Phi^{(p)}(qT') = (X^{(p)})^{-1}(qT') = e^{\sum_{k=1}^{n} -z_k^{(p)}(qT')A_k - Z_0^{(p)}}, \tag{20}$$

$$\begin{aligned} z_1^{(1)}(qT') &= \omega_2 qT'\mu, \\ z_2^{(1)}(qT') &= -\omega_1 qT'(1-\mu), \\ z_3^{(1)}(qT') &= 0, \end{aligned} \tag{21}$$

$$\begin{aligned} z_1^{(2)}(qT') &= \omega_2 qT'\mu, \\ z_2^{(2)}(qT') &= -\omega_1 qT'(1-\mu), \\ z_3^{(2)}(qT') &= -\frac{1}{2}\omega_1\omega_2 qT'^2\mu(1-\mu). \end{aligned} \tag{22}$$

4. Controllability

Towards the goal of controlling the state transition matrix $\Phi(t)$, we examine the controllability of the system

$$\dot{\Phi} = (A + Bu)\Phi \tag{23}$$

on the matrix Lie group G with Lie algebra $\mathcal{G}$. Following Jurdjevic and Sussmann [9], we make the following definitions. $\mathcal{U}$ is the class of admissible controls where $\mathcal{U}$ is either $\mathcal{U}_u$, $\mathcal{U}_r$ or $\mathcal{U}_b$ and

(i) $\mathcal{U}_u$ is the class of locally bounded and measurable functions on $[0,\infty)$ taking values in $\Re^m$.

(ii) $\mathcal{U}_r \subset \mathcal{U}_u$ with elements taking values in the unit m-dimensional cube.

(iii) $\mathcal{U}_b$ is the class of piecewise constant functions on $[0,\infty)$ taking values in $\Re^m$ where components of its elements take values in the set $\{-1,1\}$.

If $u \in \mathcal{U}$ and $\Phi_0 \in G$, we denote the solution Φ of (23) which satisfies $\Phi(0) = \Phi_0$ by $\pi(\Phi_0, u, \cdot)$, i.e., $\Phi(t) = \pi(\Phi_0, u, t)$, $\forall t \in [0,\infty)$. If $\pi(\Phi_0, u, t) = \Phi_1$ for some $t \ge 0$ then we say u steers Φ_0 into Φ_1 in t units of time. System (23) is said to be *controllable* if there exists an admissible control $u \in \mathcal{U}$ that steers any $\Phi_0 \in G$ into any $\Phi_1 \in G$, with no constraints on how many units of time are required. Define L to be the Lie algebra generated by A and B. Let $L(G)$ be the Lie algebra associated with G.

Theorem 2 (Jurdjevic and Sussmann [9]). Let G be compact and connected. System (23) with $u \in \mathcal{U}$ is controllable if and only if $L = L(G)$. Further, $\exists t' > 0$ such that for every $\Phi_0, \Phi_1 \in G$ there is a control $u \in \mathcal{U}$ that steers Φ_0 into Φ_1 in less than t' units of time. If G is semisimple, then $\exists t' > 0$ such that for every $\Phi_0, \Phi_1 \in G$ there is a control $u \in \mathcal{U}$ that steers Φ_0 into Φ_1 in exactly t' units of time.

In order to accommodate switching controls, as shown in Figure 2, we define the class of controls $\mathcal{U}_s$ as

(iv) $\mathcal{U}_s$ is the class of piecewise constant functions on $[0,\infty)$ taking values in $\Re^m$ where components of its elements take values in the set $\{0,1\}$.

Corollary 2. Theorem 2 holds when $\mathcal{U}$ is replaced by $\mathcal{U}_s$.

Proof. Let $w = 2u - 1$. Thus, if $u \in \mathcal{U}_s$ then $w \in \mathcal{U}_b$. Since $u = \frac{1}{2}(w+1)$, system (23) can be rewritten as

$$\dot{\Phi} = (A + B(\frac{1}{2}(w+1)))\Phi = ((A + \frac{1}{2}B) + \frac{1}{2}Bw)\Phi. \tag{24}$$

By Theorem 2 system (24) is controllable with $w \in \mathcal{U}_b$ if and only if the Lie algebra L_s generated by $\{(A + \frac{1}{2}B), \frac{1}{2}B\}$ is equal to $L(G)$. Since L_s is L, system (24) with $w \in \mathcal{U}_b$ is controllable if and only if $L = L(G)$. But, system (24) with $w \in \mathcal{U}_b$ is equivalent to system (23) with $u \in \mathcal{U}_s$. The remaining results follow from Theorem 2. □

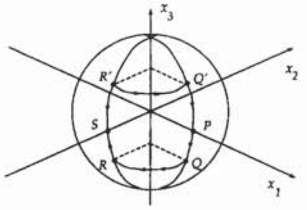

Figure 4: Paths in State Space

For our example network (Figure 1), the state transition matrix, which satisfies equation (18), evolves on the Lie group $G = SO(3)$ which is connected, compact and semisimple. $A_1 = \hat{e}_1$, $A_2 = \hat{e}_2$ and the Lie algebra L generated by $\{A_1, A_2\}$ is $so(3)$. So $L = L(SO(3))$ and by Corollary 2 the system described by (18) is controllable. Further, there exists a time $t' > 0$ such that for every $\Phi_0, \Phi_1 \in SO(3)$, there is a control $u \in \mathcal{U}_s$ that steers Φ_0 into Φ_1 in exactly t' units of time.

The system (18), however, is not controllable in arbitrarily short periods of time. Further, in order to go from some $\Phi_0 \in SO(3)$ to some other arbitrary $\Phi_1 \in SO(3)$, Φ may have to travel far. This implies that we may not be able to choose switching controls to follow any arbitrary path in the system state space.

A qualitative picture of the paths that we can follow closely can be deduced from the first-order average formula $\Phi^{(1)}$ for Φ given in equations (20) and (21). In Figure 4 we show the state space for our system, i.e., a sphere in $\Re^3$. A curve $\Phi(t)$ corresponds to a rotation of the sphere. While the $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ axes stay fixed, the system state $x(t)$ rotates with the sphere. Equation (20) can be interpreted as an Euler parametrization of the rotation $\Phi^{(1)}$ which approximates the rotation Φ. Let $z^{(1)} = (z_1^{(1)}, z_2^{(1)}, z_3^{(1)})^T$ and note that $z^{(1)}(qT') = qz^{(1)}(T')$. Let $\phi = \|z^{(1)}(T')\|$. The magnitude of the rotation $\Phi^{(1)}$ is $q\phi$ and the axis of rotation is $-z^{(1)}(T')/\phi$. Since μ is defined such that it satisfies $0 \le \mu \le 1$, equations (20) and (21) imply that $-z_1^{(1)}(T') \le 0$, $-z_2^{(1)}(T') \ge 0$ and $z_3^{(1)}(T') = 0$, i.e., the axis of rotation will always point into the quadrant of the $\mathbf{x_1}$-$\mathbf{x_2}$ plane corresponding to $x_1 \le 0$, $x_2 \ge 0$. For example, if $\mu = 0$ then $\Phi^{(1)}$ corresponds to a positive rotation about the $\mathbf{x_2}$-axis. If $\mu = 1$ then $\Phi^{(1)}$ corresponds to a negative rotation about the $\mathbf{x_1}$-axis.

In general, this type of restriction means that a path that can be followed closely in one direction cannot be followed closely in the opposite direction. In the next section we will illustrate the design of a controller that transfers energy in the example network from C_1 to C_2 following the path $PQRS$ shown in Figure 4. The path from P to Q can be achieved using $u = 0$ (i.e., $\mu = 0$). Similarly, the path from R to S can be achieved using $u = 1$ (i.e., $\mu = 1$). The path from Q to R can be followed by successive rotations about different vectors pointing into the second quadrant of the $\mathbf{x_1}$-$\mathbf{x_2}$ plane. The reverse path $SRQP$ cannot be followed closely due to the limitation on the direction of achievable rotations. Similarly, the path $SR'Q'P$ is achievable while $PQ'R'S$ is not.

5. Controller Design

In order to design a controller for the example network of Figure 1, we first develop a technique for computing the control switchings that will drive the state $x(t)$ from some initial target point x_a to the next target point x_b. We assume that the path from x_a to x_b is in the "achievable" direction as discussed in Section 4. Let the switching period T' be fixed at $T' = \epsilon$. Then the problem becomes one of choosing μ (the duty ratio) and q (the number of switching periods) such that for some initial time t_0, $x(t_0) = x_a$ and $x(t_0 + qT') = x_b$.

Using our second-order average formula $\Phi^{(2)}$ ((20) and (22)), we know the state transition matrix solution as a function of μ and q with $O(\epsilon^2)$ accuracy. To determine the desired motion of the state transition matrix, we parametrize the various rotations that will take x_a into x_b as a function of a single parameter. One rotation that will take x_a into x_b can be computed as

$$\Phi_c \triangleq e^{\theta \hat{x}_c}, \; \theta = \cos^{-1}(x_a^T x_b), \;\; x_c = \frac{x_a \times x_b}{\|x_a \times x_b\|} = \frac{\hat{x}_a x_b}{\|\hat{x}_a x_b\|}. \tag{25}$$

After this rotation has been performed, the state is invariant to rotations about the vector x_b. A class of rotations Φ_{θ_b} that will take x_a into x_b is described by

$$\Phi_{\theta_b} \triangleq e^{\theta_b \hat{x}_b} e^{\theta \hat{x}_c}, \tag{26}$$

where θ_b is a free parameter. Our goal is then to compute μ, q and θ_b such that $\Phi^{(2)}(qT') = \Phi_{\theta_b}$. We describe one method, based on quaternions, that produces a set of nonlinear algebraic equations to be solved for the exact values of μ, q and θ_b.

Method. A unit quaternion is a four-tuple that can be used as a representation of a rotation in $\Re^3$. Let $Q_c = (q_{c_0}, q_{c_1}, q_{c_2}, q_{c_3})$, $Q_b = (q_{b_0}, q_{b_1}, q_{b_2}, q_{b_3})$, $Q_t = (q_{t_0}, q_{t_1}, q_{t_2}, q_{t_3})$, $Q_z = (q_{z_0}, q_{z_1}, q_{z_2}, q_{z_3})$ be the quaternion representations of Φ_c, $\Phi_b = e^{\theta_b \hat{x}_b}$, $\Phi_{\theta_b} = \Phi_b \Phi_c$, $\Phi^{(2)}(qT')$, respectively. Then, by definition

$$\begin{aligned}
Q_c &= (\cos\frac{\theta}{2}, x_{c_1}\sin\frac{\theta}{2}, x_{c_2}\sin\frac{\theta}{2}, x_{c_3}\sin\frac{\theta}{2}),\\
Q_b &= (\cos\frac{\theta_b}{2}, x_{b_1}\sin\frac{\theta_b}{2}, x_{b_2}\sin\frac{\theta_b}{2}, x_{b_3}\sin\frac{\theta_b}{2}),\\
Q_z &= (\cos\frac{\phi}{2}, -\frac{z_1^{(2)}}{\phi}\sin\frac{\phi}{2}, -\frac{z_2^{(2)}}{\phi}\sin\frac{\phi}{2}, -\frac{z_3^{(2)}}{\phi}\sin\frac{\phi}{2}),
\end{aligned}$$

where $\phi = \|z^{(2)}(qT')\|$ and $z_i^{(2)} = z_i^{(2)}(qT')$. By the rules of quaternion multiplication, $Q_t = Q_b Q_c$ is computed as

$$Q_t = \begin{pmatrix} q_{t_0} \\ q_{t_1} \\ q_{t_2} \\ q_{t_3} \end{pmatrix} = \begin{pmatrix} q_{b_0}q_{c_0} - q_{b_1}q_{c_1} - q_{b_2}q_{c_2} - q_{b_3}q_{c_3} \\ q_{b_0}q_{c_1} + q_{b_1}q_{c_0} + q_{b_2}q_{c_3} - q_{b_3}q_{c_2} \\ q_{b_0}q_{c_2} + q_{b_2}q_{c_0} + q_{b_3}q_{c_1} - q_{b_1}q_{c_3} \\ q_{b_0}q_{c_3} + q_{b_3}q_{c_0} + q_{b_1}q_{c_2} - q_{b_2}q_{c_1} \end{pmatrix}.$$

To find μ, q and θ_b such that $\Phi^{(2)}(qT') = \Phi_{\theta_b}$, we solve $Q_t = Q_z$.

Example Problem. We illustrate a controller design example where it is desired to transfer energy from C_1 to C_2 while maintaining a constant current I_3 through the inductor, i.e., suppose we would like to drive the state $x(t)$ along the path $PQRS$ as shown in Figure 4. Let $x^T x = 1$, i.e., assume that $x(t)$ evolves on the unit sphere. Then we want $x(0) = P = (1,0,0)^T$ and $x(t_f) = S = (0,-1,0)^T$. Let us, for example, choose the constant current such that $x_3 = -1/\sqrt{2}$. Then for some $0 < s_0 < s_f < t_f$, we want $x(s_0) = Q = (1/\sqrt{2}, 0, -1/\sqrt{2})^T$ and $x(s_f) = R = (0, -1/\sqrt{2}, -1/\sqrt{2})^T$. We choose six target points $x_1, \ldots, x_6$ along the path from $x_0 = Q$ to $x_7 = R$ as

$$x_i = e^{-\frac{1}{7}\frac{\pi}{2}A_3} x_{i-1}, \quad i = 1, \ldots, 6.$$

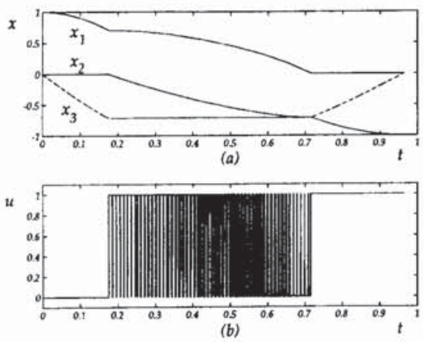

Figure 5: Control Signal and State Response

We assume the values for the network components are $C_1 = 0.1$, $C_2 = 0.2$, $L_3 = 0.5$. We note that the path from P to Q is a rotation of $\cos^{-1}(1/\sqrt{2}) = \pi/4$ radians about the $\mathbf{x_2}$-axis. We can follow this path by setting $u = \mu = 0$ for $t \in [0, s_0]$ where s_0 is computed from

$$\omega_1 s_0(1-\mu) = \omega_1 s_0 = \pi/4.$$

Similarly, the path from R to S is a rotation of $-\pi/4$ radians about the $\mathbf{x_1}$-axis. This path can be followed by setting $u = \mu = 1$ for $t \in [s_f, t_f]$ where $t_f - s_f$ is computed from

$$-\omega_2(t_f - s_f)\mu = -\omega_2(t_f - s_f) = -\pi/4.$$

To compute the switching control for the path QR, we apply the quaternion method to each of the seven steps to be taken between Q and R. Let μ_{ij} and q_{ij} denote the values of μ and q, respectively, to be used to drive $x(t)$ from x_i to x_j. Using $\epsilon = 0.01$, our method (solved numerically using MATLAB) produces

ij	01	12	23	34	45	56	67
μ_{ij}	0.93	0.80	0.69	0.59	0.47	0.33	0.14
q_{ij}	7.58	8.34	8.67	8.57	8.02	7.07	5.76

Thus, $s_f = s_0 + \sum_{i=0}^{6} q_{i(i+1)}\epsilon = s_0 + 54.0078\epsilon$. We note that the values of q_{ij} are not integers. We account for that by using a fractional period (i.e., a scaled value for T') for the last switching of each step.

The control u computed for the complete path $PQRS$ is shown in Figure 5(b). This figure shows that there are a finite number of control switchings. The duty ratio is close to 1 towards the beginning and close to 0 towards the end. Figure 5(a) shows the response of the state $x(t)$ as a function of time. The desired energy transfer can be observed, i.e., x_1 goes from 1 to 0 and x_2 goes from 0 to -1. x_3 ramps to $-1/\sqrt{2}$, stays there and then ramps back to 0 as desired. The simulation was performed using MATLAB. We show a magnified plot of $x_3(t)$ during the path QR in Figure 6. The root-mean-square error in $x_3(t)$ during the path QR when it was intended to be constant at $x_3(t) = -1/\sqrt{2}$ was computed to be $0.0037 = 37\epsilon^2$. This is consistent with the $O(\epsilon^2)$ accuracy that we expect.

6. Conclusions

The theory of averaging on Lie groups has been applied to a type of control problem in switched electrical networks. In particular, we have derived a (primarily) open-loop strategy to control energy transfers between dynamic storage elements for a switched electrical network that models the conversion portion of a dc-dc converter. The number of control switchings is predictable and finite, and the accuracy of the response is a function of the choice of switching frequency and averaging order.

The state transition matrix for the example network evolves on the matrix Lie group $SO(3)$. The network that is the topological dual [3] to this example is similarly described by a state transition matrix that evolves on $SO(3)$. Accordingly, the methodology described here can be used to control energy transfers in the dual network. Similarly, it is expected that one could extend the methodology to other more complicated networks such as those with a state transition matrix evolving on $SO(k)$, $k > 3$. The major difficulty with such an extension would be the parametrization of the rotations in $SO(k)$ that drive the state from one point to another on the sphere in $\Re^k$.

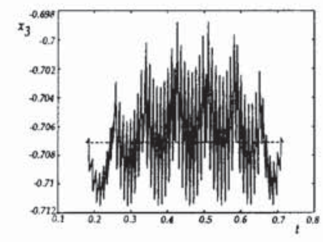

Figure 6: Response of Current Variable x_3

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