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High-Order Averaging on Lie Groups and Control of an Autonomous Underwater Vehicle [1]

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Abstract

In this paper, extending our previous work on averaging on Lie groups, we present a third-order averaging theorem for periodically forced, drift-free, left-invariant systems on Lie groups and use it to demonstrate constructive controllability for a class of problems. Specifically, this class includes the case for which depth-two Lie brackets are needed for complete controllability. We illustrate this via an example on the group $SE(3)$, appropriate as a model of kinematic control of an underwater vehicle.

1. Introduction

Drift-free systems with fewer controls than state variables arise in a variety of control problems including motion planning for wheeled robots subject to nonholonomic constraints, spacecraft attitude control and the motion control of autonomous underwater vehicles. Recent research has focused on the problem of constructing controls to achieve complete controllability [1, 2, 3, 4, 5, 6]. In particular, constructive procedures based on periodically time-varying controls have proven successful [3, 4, 5, 6].

Our interest in this paper is in constructive controllability using periodic forcing of drift-free, left-invariant systems of the form

$$\dot{X} = \epsilon X U, \qquad U(t) = \sum_{i=1}^{n} A_i u_i(t), \qquad (1)$$

evolving on matrix Lie groups. Here $X(t)$ is a curve in a matrix Lie group G of dimension n, $U(t)$ is a curve in the Lie algebra $\mathcal{G}$ of G, and $\{A_1, \ldots, A_n\}$ a basis for $\mathcal{G}$. The $u_i(\cdot)$ are assumed to be periodic functions of common period T. $\epsilon > 0$ is a small parameter such that $\epsilon u_i(\cdot)$ are interpreted as the small-amplitude, periodic controls, although some of the $u_i(\cdot)$ may be identically zero.

We state formally the complete constructive controllability problem for system (1) where $u_i(t) \equiv 0$, $i = m+1, \ldots, n$:

(P) Given an initial condition $X_i \in G$, a final condition $X_f \in G$ and a time $t_f > 0$, find $u(t) = (u_1(t), \ldots, u_m(t))$, $t \in [0, t_f]$, such that $X(0) = X_i$ and $X(t_f) = X_f$.

Here and in previous papers, to solve problem **(P)**, we use averaging theory for systems of the form (1) as a means to specify open-loop, periodic control. The goal of averaging in this context is to describe an approximate solution to (1) that evolves on the matrix group G and remains close to the actual solution, but gives rise to straightforward procedures for achieving complete constructive controllability.

First and second-order averaging theorems have been proved for systems of the form (1) [6]. In particular, the second-order average approximation provides a formula for achieving complete constructive controllability using fewer than n periodic controls if the controllability Lie algebra rank condition is satisfied for a system of the form (1) using up to depth-one Lie brackets (i.e., single brackets). In this case the formula solves **(P)** with $O(\epsilon^2)$ accuracy which could be improved with intermittent feedback if desired. Additionally, the second-order average approximation admits a geometric interpretation as an area rule [6].

In this paper we prove a third-order averaging theorem for systems of the form (1) and develop the asso-

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ciated geometric interpretation. This facilitates the design of open-loop controls to solve **(P)** with $O(\epsilon^3)$ accuracy for systems (1) which require up to depth-two Lie brackets (i.e., double brackets) to satisfy the controllability Lie algebra rank condition. As an illustration of our results, we show how to steer an autonomous underwater vehicle ($G = SE(3)$, the group of rigid motions) to a desired position and orientation when only three controls are available (two rotational and one translational). In the special case of a micro-scale underwater vehicle or a small vehicle in a highly viscous fluid, (thus implying a low Reynolds number), control of angular and translational velocities can be effected simply by cyclic body deformations, cf. [7, 8]. In imitation of the flagella or cilia used by microorganisms for maneuvering, actuators such as flapping flexible oars or rotating corkscrews could be used to generate angular and translational velocities for such a vehicle.

In the present paper, we only require control authority over two angular velocities and one translational velocity to translate and orient an underwater vehicle as desired. The low number of controls required to achieve complete constructive controllability provides a measure of redundancy to the control system. This redundancy can also be interpreted as the means for the controller to "adapt" to a failure in the system that reduces the control authority, by continuing to provide complete control over the position and orientation of the vehicle.

In Section 2 we give preliminaries, and in Section 3 we summarize first and second-order averaging. In Section 4 we prove a third-order averaging theorem and the associated constructive controllability result. We apply these results to the underwater vehicle control problem in Section 5.

2. Preliminaries

Assume that $u(t)$ is periodic in t with period T. Let $u_{av} = (u_{av1}, \ldots, u_{avn})^T$ and $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T$. Define

$$u_{avi} = \frac{1}{T}\int_0^T u_i(\tau)d\tau, \quad \tilde{u}_i(t) = \int_0^t u_i(\tau)d\tau. \quad (2)$$

$$Area_{ij}(T) = \frac{1}{2}\int_0^T (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) - \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma))d\sigma. \quad (3)$$

$$\begin{aligned} a_{ij}(t) &= \tfrac{1}{2}\textstyle\int_0^t(\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) - \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma))d\sigma. \\ &= \tfrac{Area_{ij}(T)t}{T} + f(t), \quad f(t+T) = f(t),\ f(0) = 0. \end{aligned} \quad (4)$$

$$m_{ijk}(T) = \frac{1}{3}\int_0^T (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) - \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma))\tilde{u}_k(\sigma)d\sigma. \quad (5)$$

For $u_{av} = 0$, $Area_{ij}(T)$ is interpreted as the area bounded by the closed curve described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period, and $m_{ijk}(T)$ is interpreted as a first moment.

Given $\{A_1, \ldots, A_n\}$, one computes the associated *structure constants* Γ_{ij}^k using the definition

$$[A_i, A_j] = \sum_{k=1}^n \Gamma_{ij}^k A_k, \quad i, j = 1, \ldots, n, \quad (6)$$

where $[\cdot,\cdot]$ is the Lie bracket on $\mathcal{G}$ defined by $[A, B] = AB - BA$. A *depth-μ Lie bracket* is defined as μ iterated brackets, e.g., $[B_\mu, [B_{\mu-1}, [\cdots, [B_1, B_0]\cdots]]]$. The *depth-two structure constants* θ_{ijk}^p associated with basis $\{A_1, \ldots, A_n\}$ are defined by

$$\theta_{ijk}^p \triangleq \sum_{l=1}^n \Gamma_{ij}^l \Gamma_{lk}^p, \ \text{ i.e., } \ [[A_i, A_j], A_k] = \sum_{p=1}^n \theta_{ijk}^p A_p. \quad (7)$$

Let

$$\begin{aligned} \mathcal{C} = \{B \mid B = [B_k, [B_{k-1}, [\cdots, [B_1, B_0]\cdots]]], \\ B_i \in \{A_1, \ldots, A_m\},\ i = 0, \ldots, k\}. \end{aligned} \quad (8)$$

By [9], for G a connected Lie group, if $\mathcal{G} = \text{span}\mathcal{C}$ then system (1) is *controllable.* We refer to this condition as the *Lie algebra controllability rank condition.*

We recall the *single exponential* local representation of X given by Magnus [10]. By Theorem III of [10], assuming a certain convergence criterion is met, the solution to (1) with $X(0) = I$ can be expressed as

$$X(t) = e^{Z(t)}, \quad (9)$$

where $Z(t) \in \mathcal{G}$ is given by the infinite series:

$$\begin{aligned} Z(t) &= \epsilon\int_0^t U(\tau)d\tau + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau \\ &+ \frac{\epsilon^3}{4}\int_0^t [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau \\ &+ \frac{\epsilon^3}{12}\int_0^t [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau + \ldots \end{aligned} \quad (10)$$

Satisfying the convergence criterion means limiting the duration of validity of the single exponential representation (see [11] for details).

Let $\hat{\Phi} : \mathcal{G} \rightarrow G$ define the mapping

$$\hat{\Phi}(Z) = e^Z. \quad (11)$$

Let $\hat{S}$ be the largest neighborhood of $0 \in \mathcal{G}$ such that $\hat{\Psi} \equiv \hat{\Phi}|_{\hat{S}} : \hat{S} \to G$ is one-to-one. Let $\hat{Q} \equiv \hat{\Psi}(\hat{S}) \subset G$. Then $\hat{\Psi} : \hat{S} \to \hat{Q}$ is a diffeomorphism and we can define a metric $\hat{d} : \hat{Q} \times \hat{Q} \to \Re_+$ by

$$\hat{d}(X,Y) = d(\hat{\Psi}^{-1}(X), \hat{\Psi}^{-1}(Y)) \qquad (12)$$

where $d : \Re^n \times \Re^n \to \Re_+$ is given by $d(\alpha,\beta) = \|\alpha - \beta\|$, and $\|\cdot\|$ is a norm on $\Re^n$.

3. First and Second-Order Averaging

The basic idea which we use for first and second-order averaging as well as for high-order averaging is to derive classical averaging theory approximations for the local representation and then transfer such estimates to the group level for solutions to (1). By the pth-order average approximation $X^{(p)}$, we mean that given a metric d on the Lie group G, $d(X(t), X^{(p)}(t)) = O(\epsilon^p)$, $\forall t \in [0, b/\epsilon]$, $b > 0$. The formula for the first-order average approximation $X^{(1)}$ was found to be [6]

$$X^{(1)}(t) = X^{(1)}(0)e^{\epsilon U_{av} t}. \qquad (13)$$

Assuming that $u_{av} = 0$, the formula for the second-order average approximation $X^{(2)}$ where $X^{(2)}(0) = X(0) = I$ is

$$X^{(2)}(t) = e^{Z^{(2)}(t)}, \;\; Z^{(2)}(t) = \sum_{k=1}^{n} z_k^{(2)}(t) A_k, \qquad (14)$$

$$z_k^{(2)}(t) = \epsilon \tilde{u}_k + \frac{\epsilon^2 t}{T} \sum_{i,j=1;i<j}^{n} Area_{ij}(T)\Gamma_{ij}^k. \qquad (15)$$

This formula is interpreted as an area rule since secular motion is proportional to the area terms $Area_{ij}(T)$. We show in [6] that if (1) satisfies the Lie algebra controllability rank condition with up to depth-$(p-1)$ Lie brackets, then the complete constructive controllability problem $(\mathbf{P})$ can be solved with $O(\epsilon^p)$ accuracy using the formulas for $X^{(p)}(t)$, $p = 1, 2$.

4. Third-Order Averaging

Higher-order average approximations to the solution to (1) naturally provide successively more information about the actual solution, $X(t)$. The nature of this information can be gleaned from the infinite series expansion of $Z(t)$ in (10). For example, the $O(\epsilon^2)$ term in (10) is a depth-one Lie bracket, and as described above the $O(\epsilon^2)$ approximation completely captures the effect of the depth-one Lie brackets in the context of controllability. It is expected that the $O(\epsilon^p)$ approximation for $p \geq 2$ of $X(t)$ will completely capture the effect of depth-$(p-1)$ Lie brackets in the context of controllability. In this section we prove this result for $p = 3$. Additionally, we show that the third-order approximation has a geometric interpretation based on a higher-order geometric object which can be described as a first moment. The first moment plays a role analogous to the role played by area in the second-order average approximation.

Theorem 1 (Area-Moment Rule). Let $\epsilon > 0$ be a small parameter. Let $D = \{Z \in \mathcal{G} \mid \|Z\| < r\} \subset \hat{S}$. Assume that $u(t) \in \Re^n$ is periodic in t with period $T > 0$ and has continuous derivatives up to fourth order for $t \in [0, \infty)$. Let $b > 0$ be such that the convergence requirement for (10) is met $\forall t \in [0, b/\epsilon]$. Let $X(t)$ be the solution to (1), with $X(0) = I$, represented by the single exponential (9). Define

$$Z^{(3)}(t) = \textstyle\sum_{p=1}^{n}(\epsilon \tilde{u}_p(t) + \epsilon^2 \sum_{i,j=1;i<j}^{n} a_{ij}(t)\Gamma_{ij}^p - \frac{\epsilon^3 t}{T}\sum_{k=1}^{n}\sum_{i,j=1;i<j}^{n} m_{ijk}(T)\theta_{ijk}^p)A_p, \qquad (16)$$

$$X^{(3)}(t) = e^{Z^{(3)}(t)}. \qquad (17)$$

If $Z^{(3)}(t) \in D$, $\forall t \in [0, b/\epsilon]$,

$$\hat{d}(X(t), X^{(3)}(t)) = O(\epsilon^3), \;\; \forall t \in [0, b/\epsilon].$$

Proof. Let $s = \epsilon t$. Then from (10),

$$\begin{aligned}\frac{dZ}{ds} &= U + \frac{\epsilon}{2}[\tilde{U}, U] + \frac{\epsilon^2}{4}[\int_0^s [\tilde{U}(\tau), U(\tau)]d\tau, U] \\ &+ \frac{\epsilon^2}{12}[\tilde{U}, [\tilde{U}, U]] + \ldots \triangleq f(s,\epsilon). \qquad (18)\end{aligned}$$

Let $Z_0(s)$, $Z_1(s)$ and $Z_2(s)$ be the solutions, respectively, to

$$\frac{dZ_0}{ds} = f(s,0) = U(s), \;\; Z_0(0) = 0, \qquad (19)$$

$$\frac{dZ_1}{ds} = \frac{\partial f}{\partial \epsilon}(s,0) = \frac{1}{2}[\tilde{U}, U](s), \;\; Z_1(0) = 0, \;\; (20)$$

$$\frac{dZ_2}{ds} = \frac{\partial^2 f}{\partial \epsilon^2}(s,0) = \frac{1}{4}[\int_0^s [\tilde{U}(\tau), U(\tau)]d\tau, U(s)] + \frac{1}{12}[\tilde{U}, [\tilde{U}, U]](s), \;\; Z_2(0) = 0. \qquad (21)$$

Then by standard perturbation theory (c.f. Theorem 7.1 [12]), if $Z_0(s) \in D$, $\forall s \in [0, b]$, then $\exists \epsilon^* > 0$ such that $\forall |\epsilon| < \epsilon^*$ (18) has the unique solution $Z(s)$ defined on $[0, b]$ such that $\forall s \in [0, b]$

$$\|Z(s) - (Z_0(s) + \epsilon Z_1(s) + \epsilon^2 Z_2(s,\epsilon))\| = O(\epsilon^3).$$

This implies that $\forall t \in [0, b/\epsilon]$

$$\|Z(t)-(Z_0(t)+\epsilon Z_1(t)+\epsilon^2 Z_2(t))\| = O(\epsilon^3), \quad (22)$$

where $Z_q(t),\ q = 0,1,2$ can be determined from (19) - (21) and the fact that $ds = \epsilon dt$. Now let $Y \triangleq \epsilon^2 Z_2$. Then

$$\|Z-(Z_0+\epsilon Z_1+Y)\| = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon], \quad (23)$$

$$\begin{aligned} \dot{Y} &= \tfrac{\epsilon^3}{4}[\textstyle\int_0^t[\tilde{U}(\tau), U(\tau)]d\tau, U(t)] \\ &+ \tfrac{\epsilon^3}{12}[\tilde{U},[\tilde{U},U]](t), \quad Y(0) = 0. \end{aligned} \quad (24)$$

Let $\bar{Y}(t)$ be the solution to

$$\begin{aligned} \dot{\bar{Y}} &= \tfrac{\epsilon^3}{4T}\textstyle\int_0^T[\int_0^\tau[\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau \\ &+ \tfrac{\epsilon^3}{12T}\textstyle\int_0^T[\tilde{U}(\tau),[\tilde{U}(\tau),U(\tau)]]d\tau, \quad \bar{Y}(0) = 0. \end{aligned} \quad (25)$$

By classical averaging (c.f. Theorem 7.4 [12]), if $\bar{Y} \in D,\ \forall t \in [0, b/\epsilon]$ and ϵ is small enough then $\|Y(t) - \bar{Y}(t)\| = O(\epsilon^3)$ on $[0, b/\epsilon]$. So by (23) and the triangle inequality

$$\|Z(t) - Z^{(3)}(t)\| = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon], \quad (26)$$

$$Z^{(3)}(t,\epsilon) \triangleq Z_0(t) + \epsilon Z_1(t) + \bar{Y}(t). \quad (27)$$

By (26) and the definition of $\hat{d}$, $\hat{d}(X(t), X^{(3)}(t)) = O(\epsilon^3)$, $\forall t \in [0, b/\epsilon]$, where $X^{(3)}$ is defined by (17).

The proof is complete if we show that $Z^{(3)}$ defined by (27) agrees with (16). By (19) $Z_0(t) = \epsilon\tilde{U}(t) = \sum_{p=1}^n \epsilon\tilde{u}_p(t)A_p$. Next we show that $\epsilon Z_1(t)$ is equivalent to the second term on the right side of (16).

$$\begin{aligned} \epsilon Z_1(t) &= \tfrac{\epsilon^2}{2}\textstyle\int_0^t[\tilde{U}, U](\sigma)d\sigma \\ &= \tfrac{\epsilon^2}{2}\textstyle\int_0^t[\sum_{i=1}^n \tilde{u}_i(\sigma)A_i, \sum_{j=1}^n \dot{\tilde{u}}_j(\sigma)A_j]d\sigma \\ &= \tfrac{\epsilon^2}{2}\textstyle\int_0^t \sum_{i,j=1;i<j}^n(\tilde{u}_i\dot{\tilde{u}}_j - \dot{\tilde{u}}_i\tilde{u}_j)(\sigma)[A_i,A_j]d\sigma \\ &= \textstyle\sum_{p=1}^n(\epsilon^2\sum_{i,j=1;i<j}^n a_{ij}(t)\Gamma_{ij}^p)A_p. \end{aligned}$$

Using integration by parts and an expansion similar to the above, the first and second terms of $\bar{Y}(t)$ from (25) can be expressed, respectively, as

$$\begin{aligned} &\tfrac{\epsilon^3 t}{4T}\textstyle\int_0^T[\int_0^\tau[\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau \\ &= -\tfrac{3}{4}\tfrac{\epsilon^3 t}{T}\textstyle\sum_{p=1}^n(\sum_{k=1}^n\sum_{i,j=1;i<j}^n m_{ijk}(T)\theta_{ijk}^p)A_p, \end{aligned}$$

$$\begin{aligned} &\tfrac{\epsilon^3 t}{12}\textstyle\int_0^T[\tilde{U}(\tau),[\tilde{U}(\tau),U(\tau)]]d\tau \\ &= -\tfrac{1}{4}\tfrac{\epsilon^3 t}{T}\textstyle\sum_{p=1}^n(\sum_{k=1}^n\sum_{i,j=1;i<j}^n m_{ijk}(T)\theta_{ijk}^p)A_p. \end{aligned}$$

□

The first term of $Z^{(3)}(t)$ (16) is an $O(\epsilon)$ periodic term. The second term is a secular term (linear in t), proportional to the area terms $Area_{ij}(T)$, with an $O(\epsilon^2)$ periodic term superimposed. The third term of (16) is a purely secular term proportional to the first moments $m_{ijk}(T)$. This interpretation makes Theorem 1 an area-moment rule.

Theorem 2. Suppose that system (1) satisfies the Lie algebra controllability rank condition with up to depth-two Lie brackets. Then the complete constructive controllability problem **(P)** can be solved with $O(\epsilon^3)$ accuracy using the formula for $X^{(3)}(t)$ given by (16) and (17).

Proof. Consider a system of the form (1) with $u_{m+1}(\cdot) = \ldots = u_n(\cdot) = 0,\ m \le n$. Without loss of generality we can assume that $X_i = I \in G$ and $X_f \in \hat{Q} \subset G$ is such that $Z_f = \hat{\Psi}^{-1}(X_f) = O(\epsilon^2)$. This is possible due to the left-invariance of the system and the fact that Theorem 1 can be applied repeatedly. Let

$$\begin{aligned} \mathcal{C} = \{C \mid C = A_p \text{ or } C = [A_i, A_j], \text{ or} \\ C = [[A_i, A_j], A_k],\ p,i,j,k = 1,\ldots,m\}. \end{aligned}$$

By hypothesis, $\mathcal{G} = \text{span}\mathcal{C}$. Therefore, since $Z_f \in \mathcal{G}$, $\exists c_p, c_{ij}, c_{ijk} \in \Re,\ p,i,j,k = 1,\ldots,m$ such that

$$\begin{aligned} Z_f &= \textstyle\sum_{p=1}^m c_p A_p + \sum_{p=1}^n(\sum_{i,j=1;i<j}^m c_{ij}\Gamma_{ij}^p \\ &+ \textstyle\sum_{k=1}^m\sum_{i,j=1;i<j}^m c_{ijk}\theta_{ijk}^p)A_p. \end{aligned} \quad (28)$$

Also, from (16) and the assumption that $u_i(\cdot) = 0$ for $i = m+1,\ldots,n$ we have that

$$\begin{aligned} Z^{(3)}(t,\epsilon) &= \textstyle\sum_{p=1}^m \epsilon\tilde{u}_p(t)A_p \\ &+ \textstyle\sum_{p=1}^n(\sum_{i,j=1;i<j}^m \epsilon^2 a_{ij}(t)\Gamma_{ij}^p \\ &- \textstyle\sum_{k=1}^m\sum_{i,j=1;i<j}^m \tfrac{\epsilon^3 t}{T}m_{ijk}(T)\theta_{ijk}^p)A_p. \end{aligned} \quad (29)$$

So if we choose $u_p(t),\ t \in [0,t_f],\ p = 1,\ldots,m$ such that

$$\epsilon\tilde{u}_p(t_f) = c_p, \quad p = 1,\ldots,m, \quad (30)$$

$$\epsilon^2 a_{ij}(t_f) = c_{ij}, \quad i,j = 1,\ldots,m, \quad \text{and} \quad (31)$$

$$-\frac{\epsilon^3 t_f}{T}m_{ijk}(T) = c_{ijk}, \quad i,j,k = 1,\ldots,m, \quad (32)$$

then from (28) and (29) $Z^{(3)}(t_f) = Z_f$. This implies that $X^{(3)}(t_f) = \hat{\Psi}(Z^{(3)}(t_f)) = \hat{\Psi}(Z_f) = X_f$. So, by Theorem 1, $\|X(t_f) - X_f\| = O(\epsilon^3)$.

It remains to show that (30)-(32) can be met. This becomes clear by recognizing the geometric meaning of the terms $a_{ij}(t)$ and $m_{ijk}(T)$. Specifically, a_{ij} can be controlled using a 1-1 resonance in frequencies of u_i and u_j while $m_{ijk}(T)$ can be controlled, using a 1-2-1 resonance in frequencies of u_i, u_j and u_k. See [11] for details. □

5. Underwater Vehicle Control

Consider an autonomous underwater vehicle and let $(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3})$ be coordinates fixed on the vehicle. Let $(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$ be inertial coordinates. Then we define $X(t) \in SE(3)$ by

$$X(t)\begin{bmatrix} \mathbf{r_i} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b_i} \\ 1 \end{bmatrix}.$$

That is, $X(t)$ describes the orientation and position of the vehicle at time t. Let e_1, e_2, e_3 be the standard Euclidean basis for $\Re^3$. Define $\hat{\ }: \Re^3 \rightarrow so(3)$ to be the standard isomorphism from $\Re^3$ into $so(3)$, the space of skew symmetric matrices. Let

$$A_i = \left\{ \begin{array}{ll} \left[\begin{array}{c|c} \hat{e}_i & 0 \\ \hline 0\,0\,0 & 0 \end{array}\right] & i = 1,2,3 \\ \left[\begin{array}{c|c} 0 & e_{i-3} \\ \hline 0\,0\,0 & 0 \end{array}\right] & i = 4,5,6. \end{array} \right.$$

Then $\{A_1, \ldots, A_6\}$ defines a basis for $\mathcal{G} = se(3)$, the Lie algebra associated with $SE(3)$. Now let $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ define the angular velocity of the vehicle and $v = (v_1, v_2, v_3)^T$ the vehicle translational velocity all with respect to $(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3})$. Then $X(t)$ satisfies

$$\dot{X} = X(\sum_{i=1}^{3} \Omega_i(t) A_i + \sum_{i=4}^{6} v_i(t) A_i). \qquad (33)$$

We assume that we can interpret $\Omega(t)$ and $v(t)$ as controls such that (33) is of the form (1). Specifically, suppose that we have four controls available, i.e., $\epsilon u_i(t) = \Omega_i(t)$, $i = 1,2,3$, $\epsilon u_4(t) = v_1(t)$ and $u_5(t) = u_6(t) = 0$. Then $n = 6$, $m = 4$ and since $[A_3, A_4] = A_5$ and $[A_4, A_2] = A_6$, the system is completely controllable with depth-one Lie brackets. However, if one of the rotational controls were to fail, the system is still controllable, using in general, depth-two Lie bracket. For example, suppose u_1, u_2 and u_4 are the controls available. Then $[A_1, A_2] = A_3$, $[A_4, A_2] = A_6$ and $[[A_1, A_2], A_4] = A_5$ show the system is controllable using one depth-two Lie bracket. Thus, in this case, the third-order average formula (and *not* the second-order average formula) provides a means to derive controls for complete control of the vehicle. In effect, one can think of the third-order average formula as providing an "adaptive" control law for translating and orienting an autonomous underwater vehicle in the event of an actuator failure.

We now illustrate how to use the third-order average formula to control an underwater vehicle, in the general case when one translational control u_4 and two angular controls u_1 and u_2 are available. From Theorem 1 we can write down the formula for the third-order approximation $X^{(3)}(t)$ to the solution $X(t) \in SE(3)$ as

$$X^{(3)}(t) = e^{Z^{(3)}(t)}, \quad Z^{(3)}(t) = \sum_{p=1}^{6} z_p(t) A_p,$$

$$\begin{aligned} z_1(t) &= \epsilon\tilde{u}_1(t) + \tfrac{\epsilon^3 t}{T} m_{122}(T), \\ z_2(t) &= \epsilon\tilde{u}_2(t) - \tfrac{\epsilon^3 t}{T} m_{121}(T), \\ z_3(t) &= \epsilon^2 a_{12}(t), \\ z_4(t) &= \epsilon\tilde{u}_4(t) - \tfrac{\epsilon^3 t}{T} m_{242}(T), \\ z_5(t) &= -\tfrac{\epsilon^3 t}{T}(m_{124}(T) - m_{241}(T)), \\ z_6(t) &= -\epsilon^2 a_{24}(t). \end{aligned} \qquad (34)$$

z_1, z_2, z_3 parametrize the orientation of the vehicle, while z_4, z_5, z_6 describe the position of the vehicle with respect to the $(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3})$ coordinate frame.

Now suppose without loss of generality that $X(0) = I$ and it is desired that $X(t_f) = X_d$ such that $Z_d = \sum_{p=1}^{n} z_{dp} A_p = \Psi^{-1}(X_d) = O(\epsilon^2)$. To solve the problem $(\mathbf{P})$ with $O(\epsilon^3)$ accuracy, we derive an algorithm such that $X^{(3)}(t_f) = X_d$ and apply Theorem 1. The algorithm uses sinusoidal controls and has been derived according to the geometric reasoning outlined in the proof of Theorem 2. The complete algorithm can be found in [13]. Here we present the algorithm for the case $z_{di} = 0$, $i = 1,2,3,4,6$ and $z_{d5} = O(\epsilon^2)$. This corresponds to desired motion in the double-bracket direction (i.e., translation in the $\mathbf{r_2}$ direction).

First, the time interval $[0, t_f]$ is divided into subintervals $[t_i, t_j]$ such that $t_1 = \frac{T}{4}$, $t_2 = t_1 + qT$, $t_3 = t_2 + \frac{3T}{4}$, $t_4 = t_3 + \frac{T}{4}$, $t_5 = t_4 + qT$, $t_6 = t_5 + \frac{3T}{4}$, $t_7 = t_6 + \frac{T}{4}$, $t_8 = t_7 + qT$, $t_9 = t_8 + \frac{3T}{4}$. q, an integer, and T are chosen so that $q \geq \frac{1}{\pi\epsilon}$ and $3(q+1)T = t_f$. We then let $\alpha = O(\epsilon)$, $\beta = O(\epsilon)$, $h = z_{d5}/\alpha\beta\pi q$, $\omega = 2\pi/T$ and define the (continuous) controls as

$$\epsilon u_1(t) = \begin{cases} \alpha\omega sin\omega t & 0 \leq t \leq t_6 \\ 0 & t_6 \leq t \leq t_9 \end{cases}$$

$$\epsilon u_2(t) = \begin{cases} 2\beta\omega sin\omega t & 0 \leq t \leq t_1 \\ 2\beta\omega cos(2\omega(t - t_1)) & t_1 \leq t \leq t_2 \\ 2\beta\omega cos(\omega(t - t_2)) & t_2 \leq t \leq t_3 \\ -2\beta\omega sin(\omega(t - t_3)) & t_3 \leq t \leq t_4 \\ -2\beta\omega cos(2\omega(t - t_4)) & t_4 \leq t \leq t_5 \\ -2\beta\omega cos(\omega(t - t_5)) & t_5 \leq t \leq t_6 \\ -2\beta\omega sin(\omega(t - t_6)) & t_6 \leq t \leq t_7 \\ -2\beta\omega cos(2\omega(t - t_7)) & t_7 \leq t \leq t_8 \\ -2\beta\omega cos(\omega(t - t_8)) & t_8 \leq t \leq t_9 \end{cases}$$

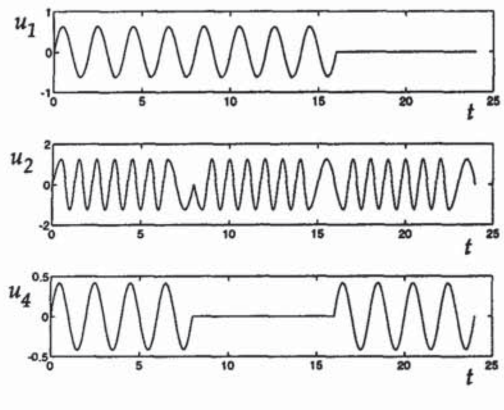

Figure 1: Control Input Signals for Example.

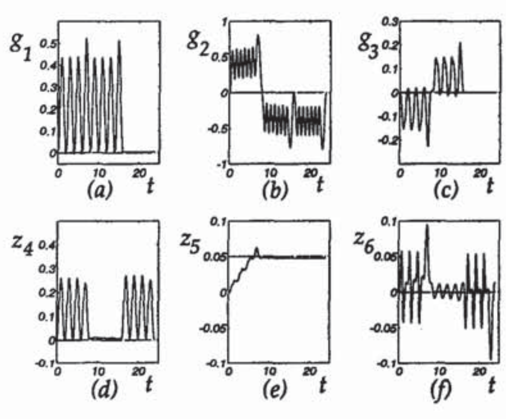

Figure 2: Response of Underwater Vehicle for Example.

$$\epsilon u_4(t) = \begin{cases} h\omega sin\omega t & 0 \leq t \leq t_3 \\ 0 & t_3 \leq t \leq t_6 \\ h\omega sin(\omega(t - t_6)) & t_6 \leq t \leq t_9 \end{cases}$$

For numerical illustration let $\epsilon = 0.2$, $z_{d5} = 0.05$ and $t_f = 24$. This represents one step in a multi-step maneuver. We choose $q = 3$, $T = 2$ and $\alpha = \beta = 0.2$ so that the frequency is relatively low, i.e., $\omega = \pi$. Figure 1 shows the three controls u_1, u_2 amd u_4, as a function of time. Figure 2 shows a simulation of the response of the system produced by MATLAB. The orientation of the vehicle is given in Figures 2(a), 2(b) and 2(c) which show plots of g_1, g_2, g_3, respectively. Here g_1, g_2, g_3 are a type of Euler angle parametrization of the orientation of the vehicle. The position of the vehicle is given in Figures 2(d), 2(e) and 2(f) which show plots of z_4, z_5, z_6, respectively. The horizontal lines represent the desired values of the parameters. It is clear, from Figure 3, that at the end of the simulation $\|X(t_f) - X_d\| = O(\epsilon^3)$, i.e., the vehicle has been moved and oriented as desired with $O(\epsilon^3)$ accuracy.

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