49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Cyclic pursuit in three dimensions

K. S. Galloway, E. W. Justh, and P. S. Krishnaprasad

***Abstract*— Pursuit strategies for interacting particles and feedback laws to execute them are formulated in three dimensions, focusing on constant bearing (CB) pursuit – a case of interest in biology. In the analysis of such laws for the setting of n particles engaged in cyclic pursuit, we reveal interesting invariant manifold dynamics and associated explicit integrability properties, as well as conditions for special solutions such as relative equilibria.**

I. Introduction

This paper explores the geometry of pursuit strategies in three dimensions, focusing on a particular class of strategies known as *constant bearing (CB) pursuit*. It also extends the program of analysis of cyclic pursuit in [2] toward design for collective behavior of n particles in three dimensions. While interest in the mathematics of pursuit has a long history (see [10]), the more recent concerns of cooperative robotics, and problem solving using collective intelligence (see introduction and references cited in [2]), have shaped the course of our research. In particular, we envision possible technological applications such as collective control of "flocks" of UAV's operating in three-dimensional space. In this paper, and in our previous work, a unifying theme centers on the idea that feedback control laws that execute pursuit strategies may serve as effective building blocks for collective behavior in nature and in machines. Our interest in CB pursuit as a strategy worthy of investigation is underscored by observations of this in nature, specifically in the high speed stoop behavior of the peregrine falcon diving from great heights to hunt prey ([8],[9]).

The constant bearing pursuit law in the planar setting [7] involves a relative bearing error and a term similar to the motion camouflage law in [3]. In section III of this paper, the appropriate extension of the cyclic CB pursuit feedback law to three dimensions is formulated (see equation 14). This leads to a derivation of an invariant manifold for CB pursuit and dynamics on this manifold (see equation 17). Section IV is devoted to the special case of $n = 2$ that we refer to as mutual CB pursuit, in analogy with mutual motion camouflage (MMC) investigated in [1]. This case reveals the presence of conservation laws leading to explicit integrability of the dynamics, a key contribution of this paper. The paper ends with conditions for existence of rectilinear and planar circling relative equilibrium motions for cyclic CB pursuit dynamics for general n.

This research was supported in part by the Air Force Office of Scientific Research under AFOSR grants FA9550710446 and FA95501010250; by the Army Research Office under ARO grant W911NF0610325; by the ODDR&E MURI2007 Program Grant N000140710734 (through the Office of Naval Research); by the NSF-NIH Collaborative Research in Computational Neuroscience Program (CRCNS2004) NIH-NIBIB grant 1 R01 EB004750-01; and by the Office of Naval Research.

K.S. Galloway and P.S. Krishnaprasad are with the Institute for Systems Research and the Department of Electrical and Computer Engineering at the University of Maryland, College Park, MD 20742, USA. kgallow1@umd.edu, krishna@umd.edu

E.W. Justh is with the Naval Research Laboratory, Washington, DC 20375, USA.

II. Modeling interactions

The evolution of a system of multiple agents moving in three-dimensional space can be described in terms of unit-mass particles tracing out twice continuously-differentiable curves, with system dynamics derived from the natural Frenet frame equations. (See, for example, [4] for details.) As in figure 2 of [4], the state of the i^{th} particle (i.e. agent) with respect to a fixed inertial frame is denoted by the position vector $\mathbf{r}_i$ and the respective natural Frenet frames $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$. If we constrain the agents to move at unit speed, then the dynamics of a system of n agents can be described by

$$\begin{aligned} \dot{\mathbf{r}}_i &= \mathbf{x}_i, \\ \dot{\mathbf{x}}_i &= u_i \mathbf{y}_i + v_i \mathbf{z}_i, \\ \dot{\mathbf{y}}_i &= -u_i \mathbf{x}_i, \\ \dot{\mathbf{z}}_i &= -v_i \mathbf{x}_i,\ i = 1, 2, \ldots, n, \end{aligned} \tag{1}$$

where u_i and v_i are the natural curvatures viewed as controls and are required to be $SE(3)$-invariant (i.e. invariant to translations and rotations of the inertial frame). We define the baseline vector $\mathbf{r}_{i,i+1} = \mathbf{r}_i - \mathbf{r}_{i+1}$, with addition in the indices interpreted as modulo n, and prohibit "sequential colocation" (i.e. we assume $|\mathbf{r}_{i,i+1}| > 0$ for all t). This means that we restrict our analysis away from the point of actual capture or rendezvous, ensuring that the control laws of section III are well-posed. Explicitly, we let the state space

$$M_{state} = \Big\{ (\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n) \mid \mathbf{r}_{i,i+1} \neq \mathbf{0},\ i = 1, 2, \ldots, n \Big\}, \tag{2}$$

where it is understood that $\mathbf{r}_i \in \mathbb{R}^3$ and that $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ are orthonormal vectors in $\mathbb{R}^3$ for each i. This paper is focused on a particular setting of cyclic pursuit (i.e. agent i pursues agent $i+1$ modulo n), defined by the constant bearing pursuit strategy in section III-B.

III. Pursuit strategies and steering laws

Steering laws for the execution of planar pursuit strategies under the framework described in section II have been developed for *classical pursuit (CP)* and constant bearing pursuit [7] as well as *motion camouflage (MC) pursuit* [3]. A three-dimensional version of the motion camouflage pursuit law is also developed in [5]. Here we derive pursuit laws for the execution of classical pursuit and constant bearing pursuit strategies in $\mathbb{R}^3$.

A. Classical Pursuit

The classical pursuit strategy specifies that the pursuer should always move directly towards the current location of the pursuee. As in the planar case [7], we define our cost function by

$$\Lambda_i^{CP} = \mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \tag{3}$$

noting that $\Lambda_i^{CP} \in [-1,1]$ and $\Lambda_i^{CP} = -1$ corresponds to attainment of the CP strategy. With the following notation

$$\bar{x}_i \triangleq \mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \quad \bar{y}_i \triangleq \mathbf{y}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \quad \bar{z}_i \triangleq \mathbf{z}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \tag{4}$$

we have,

Proposition 1: Consider a two-particle system in which (u_2, v_2) are arbitrary (but continuous and bounded) and (u_1, v_1) are prescribed by

$$\begin{aligned} u_1 &= -\mu \bar{y}_1 - \frac{1}{|\mathbf{r}|}\left[\mathbf{z}_1 \cdot \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right] \\ v_1 &= -\mu \bar{z}_1 + \frac{1}{|\mathbf{r}|}\left[\mathbf{y}_1 \cdot \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right], \end{aligned} \tag{5}$$

where $\mu > 0$ is a control gain and $\mathbf{r} \triangleq \mathbf{r}_1 - \mathbf{r}_2$. Then under the closed-loop dynamics (1), $\dot{\Lambda}_1^{CP} \le 0$ with $\dot{\Lambda}_1^{CP} = 0$ if and only if $\Lambda_1^{CP} = \pm 1$.

Proof: We proceed by differentiating Λ_1^{CP} along trajectories of the closed loop dynamics. First, note that

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right) = \frac{\mathbf{w}}{|\mathbf{r}|}, \tag{6}$$

where $\mathbf{w}$, the transverse component of the relative velocity, is defined by

$$\mathbf{w} = \dot{\mathbf{r}} - \left(\dot{\mathbf{r}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\right)\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{|\mathbf{r}|} \times \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right). \tag{7}$$

(See [5] and [7].) Then differentiating Λ_1^{CP} we have

$$\begin{aligned} \dot{\Lambda}_1^{CP} &= u_1 \bar{y}_1 + v_1 \bar{z}_1 + \frac{1}{|\mathbf{r}|}(\mathbf{x}_1 \cdot \mathbf{w}) \\ &= -\mu \bar{y}_1^2 - \mu \bar{z}_1^2 + \frac{1}{|\mathbf{r}|}(\mathbf{x}_1 \cdot \mathbf{w}) \\ &\quad - \frac{1}{|\mathbf{r}|}\left\{\left[\mathbf{z}_1 \cdot \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right]\bar{y}_1 - \bar{z}_1\left[\mathbf{y}_1 \cdot \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right]\right\} \\ &= -\mu\left(1 - \left(\Lambda_1^{CP}\right)^2\right), \end{aligned} \tag{8}$$

where we have progressed from the second equality to the third equality by writing out the full expressions for $\bar{y}_1$ and $\bar{z}_1$ and using the identity $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ for arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. The claims of **Proposition 1** readily follow from (8). $\square$

B. Definition of the Constant Bearing Pursuit strategy

In the planar case, the notion of constant bearing strategy simply extends CP by specifying a fixed, possibly nonzero angle between pursuer heading and the relative location of the target. The following specifies an extension of this idea to three dimensions.

Definition 1: Given a two-particle system with dynamics (1) and a parameter $a_1 \in [-1,1]$, we say particle 1 has attained the CB(a_1) pursuit strategy if $\mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = a_1$.

Remark: Given a scalar parameter $a \in [-1,1]$ and an arbitrary unit vector $\mathbf{q}$ regarded as a point on the unit sphere S^2, the set $\left\{\mathbf{y} \in S^2 \,\middle|\, \mathbf{q} \cdot \mathbf{y} = a\right\}$ defines a *small circle* (i.e. the intersection of a sphere with a plane that does not pass through the center of the sphere)[1]. Since $\mathbf{x}_1$ and $\frac{\mathbf{r}}{|\mathbf{r}|}$ are both unit vectors, we can think of the CB(a_1) pursuit strategy as prescribing a small circle centered around the point $\frac{\mathbf{r}}{|\mathbf{r}|} \in S^2$. CB($a_1$) pursuit holds when $\mathbf{x}_1$ lies on that small circle.

Remark: Observe that this definition of the three-dimensional CB pursuit strategy is fundamentally different from the planar version presented in [7] (i.e. $R(\alpha)\mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = -1$, where $R(\alpha)\mathbf{x}_1$ is the vector $\mathbf{x}_1$ rotated counterclockwise in the plane by the angle α) in that the planar version prescribed not only a constant bearing angular offset but also a particular direction (i.e. counterclockwise) for the offset. We can relate the CB strategy presented here to the planar strategy as follows. Given unit vectors $\mathbf{x}_1$ and $\frac{\mathbf{r}}{|\mathbf{r}|}$ in the plane and the two statements $R(\alpha)\mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = -1$ and $\mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = a$, we seek to define the relationship between α and a. If we define θ as the signed angle (CCW rotation positive) from $\mathbf{x}_1$ to $\frac{\mathbf{r}}{|\mathbf{r}|}$, then $\cos\theta = a$ and $|\theta - \alpha| = \pi$, i.e.

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha = -1. \tag{9}$$

This relationship holds only if $(\cos\alpha, \sin\alpha) = -(\cos\theta, \sin\theta)$, and since $\cos\theta = a$ and $\sin\theta = \pm\sqrt{1-a^2}$, the two discrete possibilities are given by $(\cos\alpha, \sin\alpha) = (-a, \mp\sqrt{1-a^2})$. Therefore the CB strategy of **Definition 1** differs from the planar strategy (in [7]) in that it allows for two discrete possibilities for pursuit geometries as opposed to the single geometry prescribed by the planar strategy.

We define a CB cost function for agent i by

$$\Lambda_i = \frac{1}{2}\left[\left(\mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\right) - a_i\right]^2 = \frac{1}{2}\left(\bar{x}_i - a_i\right)^2, \tag{10}$$

with $0 \le \Lambda_i \le \max\left[\frac{1}{2}(1-a_i)^2, \frac{1}{2}(-1-a_i)^2\right]$. Then the CB pursuit strategy of **Definition 1** is equivalent to $\Lambda_i = 0$.

Remark: At first glance, it may appear that a viable alternative definition for the three-dimensional CB pursuit strategy is obtained by letting $\tilde{\Lambda} \triangleq B\mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$, where $B \in SO(3)$ (the rotation group in three dimensions), and then defining the CB pursuit strategy by $\tilde{\Lambda} = -1$. This definition is appealing

[1]More precisely, the set $\left\{\mathbf{y} \in S^2 \,\middle|\, \mathbf{q} \cdot \mathbf{y} = a\right\}$ describes a small circle only if $a \neq 0$. For $a = 0$, it defines a great circle.

since it is the obvious extension of the previously mentioned planar CB pursuit strategy. However, a few straightforward calculations reveal that $\tilde{\Lambda}$ is not invariant to rotations of the coordinate frame (i.e. not $SO(3)$-invariant) and therefore all associated pursuit laws will be inadmissible under our framework (unless B is the identity matrix).

C. A feedback law for CB Pursuit

Proposition 2: Consider a two-particle system in which (u_2, v_2) are arbitrary (but continuous and bounded) and (u_1, v_1) are prescribed by

$$\begin{aligned} u_1 &= -\mu_1\Big(\bar{x}_1 - a_1\Big)\bar{y}_1 - \frac{1}{|\mathbf{r}|}\left[\mathbf{z}_1 \cdot \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right] \\ v_1 &= -\mu_1\Big(\bar{x}_1 - a_1\Big)\bar{z}_1 + \frac{1}{|\mathbf{r}|}\left[\mathbf{y}_1 \cdot \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right], \end{aligned} \quad (11)$$

where $\mu_1 > 0$ is a control gain. Then under the closed-loop dynamics (1), $\dot{\Lambda}_1 \le 0$ with $\dot{\Lambda}_1 = 0$ if and only if $\Lambda_1 = 0$ or $\mathbf{x}_1 \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \pm 1$.

Proof: By a series of calculations analogous to the derivation of (8), it is possible to show that

$$\dot{\Lambda}_1 = -\mu_1 \left(\bar{x}_1 - a_1\right)^2 \left(1 - \bar{x}_1^2\right) = -2\mu_1\Lambda_1\left(1 - \bar{x}_1^2\right), \quad (12)$$

from which the result follows. □

D. An invariant submanifold for cyclic CB pursuit

As in previous work on planar cyclic pursuit [2], we define the submanifold of system states for which each agent i pursues agent $(i+1)$ modulo n with a pursuit law of the form (11), and all agents have attained CB pursuit. Since $\Lambda_i = 0$ if and only if agent i has attained CB pursuit of agent $(i+1)$, we define the submanifold $M_{CB(\mathbf{a})} \subset M_{state}$ by

$$M_{CB(\mathbf{a})} = \Big\{(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n) \in M_{state} \mid \Lambda_i = 0,\ i = 1, 2, \ldots, n\Big\}, \quad (13)$$

where $\mathbf{a} = \{a_1, a_2, \ldots, a_n\}$. It follows from an argument analogous to that in **Proposition 2** that $M_{CB(\mathbf{a})}$ is an invariant manifold under cyclic pursuit dynamics (in the sense that the closed-loop vector field is tangent to the manifold). In the following proposition we prove asymptotic convergence to $M_{CB(\mathbf{a})}$ under suitable conditions.

Proposition 3: Consider the n-particle cyclic CB pursuit system governed by the closed-loop dynamics (1) with curvature controls for the i^{th} agent prescribed by

$$\begin{aligned} u_i &= -\mu_i\left(\bar{x}_i - a_i\right)\bar{y}_i - \frac{1}{|\mathbf{r}_{i,i+1}|}\left[\mathbf{z}_i \cdot \left(\dot{\mathbf{r}}_{i,i+1} \times \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\right)\right] \\ v_i &= -\mu_i\left(\bar{x}_i - a_i\right)\bar{z}_i + \frac{1}{|\mathbf{r}_{i,i+1}|}\left[\mathbf{y}_i \cdot \left(\dot{\mathbf{r}}_{i,i+1} \times \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\right)\right], \end{aligned} \quad (14)$$

where $\mu_i > 0$ and we assume $a_i \neq \pm 1$. Define the set

$$\Omega = \Big\{(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n) \in M_{state} \mid \Lambda_i \le -\epsilon + \min\left[\frac{1}{2}(-1 - a_i)^2, \frac{1}{2}(1 - a_i)^2\right], \ i = 1, 2, \ldots, n\Big\} \quad (15)$$

for $0 < \epsilon \ll \min_{i \in \{1,2,\ldots,n\}} \frac{1}{2}(\pm 1 - a_i)^2$. Then any bounded trajectory starting in Ω converges to $M_{CB(\mathbf{a})}$.

Proof: Note that Ω is closed (but not necessarily bounded) and excludes states for which $\bar{x}_i = \pm 1$ for any i. Making use of (10) we define $\Lambda = \sum_{i=1}^{n} \Lambda_i$, observing (from (12)) that

$$\dot{\Lambda} = -2\sum_{i=1}^{n} \mu_i \Lambda_i \left(1 - \bar{x}_i^2\right) \quad (16)$$

and therefore $\dot{\Lambda} \le 0$ on Ω with $\dot{\Lambda} = 0$ on Ω if and only if $\Lambda_i = 0,\ i = 1, 2, \ldots, n$. The hypothesis of boundedness of the trajectory ensures by Birkhoff's theorem the ω-limit set is nonempty, compact and invariant. Asymptotic convergence to $M_{CB(\mathbf{a})}$ follows as in the steps in the proof of LaSalle's Invariance Principle [6]. □

Note that on $M_{CB(\mathbf{a})}$ the terms of the controls (14) which involve the gains μ_i are identically zero, and therefore we can formulate reduced (closed-loop) dynamics on $M_{CB(\mathbf{a})}$ for $i = 1, 2, \ldots, n$ as

$$\begin{aligned} \dot{\mathbf{r}}_i &= \mathbf{x}_i, \\ \dot{\mathbf{x}}_i &= \frac{-1}{|\mathbf{r}_{i,i+1}|}\Bigg[\Big(\mathbf{z}_i \cdot \Big(\dot{\mathbf{r}}_{i,i+1} \times \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\Big)\Big)\mathbf{y}_i - \Big(\mathbf{y}_i \cdot \Big(\dot{\mathbf{r}}_{i,i+1} \times \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\Big)\Big)\mathbf{z}_i\Bigg] \\ \dot{\mathbf{y}}_i &= \frac{1}{|\mathbf{r}_{i,i+1}|}\left[\mathbf{z}_i \cdot \left(\dot{\mathbf{r}}_{i,i+1} \times \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\right)\right]\mathbf{x}_i, \\ \dot{\mathbf{z}}_i &= \frac{-1}{|\mathbf{r}_{i,i+1}|}\left[\mathbf{y}_i \cdot \left(\dot{\mathbf{r}}_{i,i+1} \times \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\right)\right]\mathbf{x}_i. \end{aligned} \quad (17)$$

IV. Mutual CB pursuit

As a first step towards understanding the behavior of our system under cyclic CB pursuit, we analyze the two-particle "mutual CB pursuit" case. (This can be compared with the analysis of "mutual motion camouflage" in [1].)

For analysis of two-particle systems in three dimensions, [4] demonstrates the utility of considering the reduced system $(\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2)$ evolving on $\mathbb{R}^3 \times S^2 \times S^2$, where $\mathbf{r} \triangleq \mathbf{r}_1 - \mathbf{r}_2$. Starting from (17), we derive the $(\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2)$ dynamics on $M_{CB(\mathbf{a})}$ by first computing

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \frac{1}{|\mathbf{r}|}\left[\mathbf{z}_1\Big(\Big(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\Big) \cdot \mathbf{y}_1\Big) - \mathbf{y}_1\Big(\Big(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\Big) \cdot \mathbf{z}_1\Big)\right] \\ &= \frac{1}{|\mathbf{r}|}\left[\mathbf{x}_1 \times \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right]. \end{aligned} \quad (18)$$

Here we have made use of the so-called BAC-CAB identity.

Doing similar computations for particle 2, we arrive at

$$\begin{aligned}\dot{\mathbf{r}} &= \mathbf{x}_1 - \mathbf{x}_2, \\ \dot{\mathbf{x}}_1 &= \frac{1}{|\mathbf{r}|}\left[\mathbf{x}_1 \times \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right] = \frac{1}{|\mathbf{r}|}\left(\mathbf{x}_1 \times \boldsymbol{\ell}\right), \\ \dot{\mathbf{x}}_2 &= \frac{1}{|\mathbf{r}|}\left[\mathbf{x}_2 \times \left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right)\right] = \frac{1}{|\mathbf{r}|}\left(\mathbf{x}_2 \times \boldsymbol{\ell}\right),\end{aligned} \tag{19}$$

with $\boldsymbol{\ell} \triangleq \dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}$ and initial conditions governed by the $M_{CB(\mathbf{a})}$ constraints.

A. Explicit solutions for system behavior on $M_{CB(\mathbf{a})}$

As an aid to intuition, we note that the dynamics of the baseline vector $\mathbf{r}$ can be reformulated as

$$\dot{\mathbf{r}} = \frac{1}{|\mathbf{r}|}\left(\dot{\mathbf{r}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\right)\mathbf{r} - \frac{1}{|\mathbf{r}|}\left(\dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}\right) \times \mathbf{r}. \tag{20}$$

(See [5] for background and a similar approach.) The first term captures the lengthening or shortening of the baseline vector $\mathbf{r}$, and the second term is related to the angular velocity of $\mathbf{r}$ (with $\mathbf{r}$ viewed as an extensible rod from the perspective of particle 1). Addressing the former term, we first note that $\dot{\mathbf{r}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{d}{dt}(|\mathbf{r}|)$. Defining $\rho \triangleq |\mathbf{r}|$, we have

$$\dot{\rho} = (\mathbf{x}_1 - \mathbf{x}_2) \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = a_1 + a_2, \tag{21}$$

and obtain

$$\rho(t) = (a_1 + a_2)\, t + \rho_0, \ \text{for } \rho_0 = |\mathbf{r}(0)|\,. \tag{22}$$

Turning to the second term in (20), by taking the derivative of the vector cross product $\boldsymbol{\ell} = \dot{\mathbf{r}} \times \frac{\mathbf{r}}{|\mathbf{r}|}$ along trajectories of (19), one can show that $\boldsymbol{\ell}$ is in fact a fixed vector. Substituting this result as well as our results from (21) and (22) into (20), we can express our $\mathbf{r}$ dynamics as

$$\dot{\mathbf{r}}(t) = \frac{1}{a_+ t + \rho_0}\left[a_+ \mathbb{1} - \hat{\boldsymbol{\ell}}\right]\mathbf{r}(t), \tag{23}$$

where we denote $a_+ = a_1 + a_2$ and make use of the operator $\hat{\ }: \mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$ which maps any 3-vector $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ to a skew-symmetric matrix defined by

$$\hat{\Gamma} = \begin{pmatrix} 0 & -\Gamma_3 & \Gamma_2 \\ \Gamma_3 & 0 & -\Gamma_1 \\ -\Gamma_2 & \Gamma_1 & 0 \end{pmatrix}. \tag{24}$$

Since $a_+\mathbb{1}$ and $\hat{\boldsymbol{\ell}}$ commute, for $a_+ \neq 0$ we can derive an explicit solution for $\mathbf{r}(t)$ by

$$\begin{aligned}\mathbf{r}(t) &= \exp\left(\int_0^t \frac{a_+}{a_+\tau + \rho_0} d\tau\right) \\ &\qquad \exp\left(-\hat{\boldsymbol{\ell}}\int_0^t \frac{1}{a_+\tau + \rho_0} d\tau\right)\mathbf{r}(0) \\ &= \frac{a_+ t + \rho_0}{\rho_0}\exp\left(-\frac{1}{a_+}\hat{\boldsymbol{\ell}}\ln\left(\frac{a_+ t + \rho_0}{\rho_0}\right)\right)\mathbf{r}(0).\end{aligned} \tag{25}$$

A straightforward calculation based on (23) easily yields the result for the $a_+ = 0$ case, and we can therefore write our complete solution as

$$\begin{aligned}\mathbf{r}(t) &= \begin{cases} \frac{a_+ t + \rho_0}{\rho_0}\exp\left(-\frac{1}{a_+}\hat{\boldsymbol{\ell}}\ln\left(\frac{a_+ t + \rho_0}{\rho_0}\right)\right)\mathbf{r}_0 & \text{for } a_+ \neq 0 \\ \exp\left(-\frac{1}{\rho_0}\hat{\boldsymbol{\ell}}t\right)\mathbf{r}_0 & \text{for } a_+ = 0, \end{cases} \\ &\text{for } \mathbf{r}(0) = \mathbf{r}_0,\ \rho_0 = |\mathbf{r}_0|\,,\ \mathbf{x}_i(0) = \mathbf{x}_i^0, \\ &\boldsymbol{\ell} = \left(\mathbf{x}_1^0 - \mathbf{x}_2^0\right) \times \frac{\mathbf{r}_0}{|\mathbf{r}_0|}.\end{aligned} \tag{26}$$

Similarly, by analogous calculations from (19) we have

$$\mathbf{x}_i(t) = \begin{cases} \exp\left(-\frac{1}{a_+}\hat{\boldsymbol{\ell}}\ln\left(\frac{a_+ t + \rho_0}{\rho_0}\right)\right)\mathbf{x}_i^0 & \text{for } a_+ \neq 0 \\ \exp\left(-\frac{1}{\rho_0}\hat{\boldsymbol{\ell}}t\right)\mathbf{x}_i^0 & \text{for } a_+ = 0. \end{cases} \tag{27}$$

B. Center of mass trajectory

Prior to stating and proving a proposition concerning the motion of the center of mass, we note the following calculation. Define $\Theta \in [-1, 1]$ as

$$\Theta \triangleq (\mathbf{x}_1 \times \mathbf{x}_2) \cdot \frac{\mathbf{r}}{|\mathbf{r}|}, \tag{28}$$

the signed volume of the parallelepiped with edges $\mathbf{x}_1, \mathbf{x}_2, \frac{\mathbf{r}}{|\mathbf{r}|}$. It can then be demonstrated that

$$\Theta = -\mathbf{x}_1 \cdot \boldsymbol{\ell} \ = \ -\mathbf{x}_2 \cdot \boldsymbol{\ell}. \tag{29}$$

By differentiating (29) along trajectories of (19), one can readily show that Θ is a constant value on $M_{CB(\mathbf{a})}$.

Proposition 4: Consider a two-particle system operating on $M_{CB(\mathbf{a})}$ according to the closed-loop mutual CB pursuit dynamics (19) with initial conditions $\mathbf{r}_i(0) = \mathbf{r}_i^0$ and $\mathbf{x}_i(0) = \mathbf{x}_i^0$, $i = 1, 2$. Define the change of coordinates $\tilde{\mathbf{r}}_i \triangleq \mathbf{r}_i - \mathbf{r}_c^0$, where $\mathbf{r}_c^0$ is defined by

$$\mathbf{r}_c^0 \triangleq \begin{cases} \mathbf{z}_0 - \sigma_0\left(\frac{\mathbf{r}_0}{|\mathbf{r}_0|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) & \text{for } \boldsymbol{\ell} \neq 0, \\ 0 & \text{for } \boldsymbol{\ell} = 0, \end{cases} \tag{30}$$

with $\mathbf{z}_0 = \frac{1}{2}\left(\mathbf{r}_1^0 + \mathbf{r}_2^0\right)$, $\sigma_0 = -\frac{(a_1 - a_2)}{2|\boldsymbol{\ell}|}\rho_0$, and $\mathbf{r}_0$, ρ_0, and $\boldsymbol{\ell}$ as in (26). Then the trajectory of the center of mass $\mathbf{z} \triangleq \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ can be given in the new coordinates $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{r}_c^0$ by the following:

(i.) if $\boldsymbol{\ell} = 0$, then $\tilde{\mathbf{z}}(t) = \tilde{\mathbf{z}}_0 + \frac{1}{2}\left(\mathbf{x}_1^0 + \mathbf{x}_2^0\right)t$

(ii.) if $\boldsymbol{\ell} \neq 0$, but $a_- = 0$, then $\tilde{\mathbf{z}}(t) = -\frac{\Theta}{|\boldsymbol{\ell}|^2}\boldsymbol{\ell}t$

(iii.) if $\boldsymbol{\ell} \neq 0$, $a_- \neq 0$, but $a_+ = 0$, then

$$\tilde{\mathbf{z}}(t) = \exp\left(-\frac{1}{\rho_0}\hat{\boldsymbol{\ell}}t\right)\tilde{\mathbf{z}}_0 - \frac{\Theta}{|\boldsymbol{\ell}|^2}\boldsymbol{\ell}t$$

(iv.) if $\boldsymbol{\ell}, a_-$ and a_+ are all nonzero, then

$$\tilde{\mathbf{z}}(t) = c(t)\exp\left(-\frac{1}{a_+}\hat{\boldsymbol{\ell}}\ln(c(t))\right)\tilde{\mathbf{z}}_0 - \frac{\Theta}{|\boldsymbol{\ell}|^2}\boldsymbol{\ell}t, \tag{31}$$

with $a_+ \triangleq a_1 + a_2$, $a_- \triangleq a_1 - a_2$, and $c(t) = \frac{a_+ t + \rho_0}{\rho_0}$.

Proof: Assume $\boldsymbol{\ell} \neq 0$. We will demonstrate that the center of mass follows either a circling, helical, or spiral trajectory centered on the point $\mathbf{r}_c^0$. We can resolve $\tilde{\mathbf{z}}$ into component vectors as

$$\tilde{\mathbf{z}} = \left(\tilde{\mathbf{z}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\right) \frac{\mathbf{r}}{|\mathbf{r}|} + \left(\tilde{\mathbf{z}} \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|} + \left[\tilde{\mathbf{z}} \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right)\right] \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right). \tag{32}$$

The main thrust of the proof is to demonstrate that the first term is identically zero, the second term is linear in t, and that self-contained dynamics (and a resulting closed-form solution) can be derived for the third term. We address the first term by defining $\gamma_1 \triangleq \left(\tilde{\mathbf{z}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\right)$ and $\gamma_2 \triangleq \dot{\gamma_1}$. By direct calculation, one can show that

$$\gamma_2 = \frac{1}{2}a_- + \frac{1}{\rho}\left(\tilde{\mathbf{z}} \cdot \dot{\mathbf{r}} - \gamma_1 a_+\right) \tag{33}$$

and that the system (γ_1, γ_2) evolves according to

$$\dot{\gamma_1} = \gamma_2; \quad \dot{\gamma_2} = -\gamma_1 \frac{|\boldsymbol{\ell}|^2}{\rho^2} - \gamma_2 \frac{a_+}{\rho}. \tag{34}$$

By (30) we have

$$\begin{aligned} \gamma_1(0) &= \sigma_0 \left(\frac{\mathbf{r}_0}{|\mathbf{r}_0|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \cdot \frac{\mathbf{r}_0}{|\mathbf{r}_0|} = 0, \\ \gamma_2(0) &= \frac{a_-}{2} + \frac{\sigma_0}{\rho_0}\left[\left(\frac{\mathbf{r}_0}{|\mathbf{r}_0|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \cdot \dot{\mathbf{r}}(0)\right] \\ &= \frac{a_-}{2} - \frac{a_-}{2|\boldsymbol{\ell}|}\left[\left(\dot{\mathbf{r}}(0) \times \frac{\mathbf{r}_0}{|\mathbf{r}_0|}\right) \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right] = 0. \end{aligned} \tag{35}$$

Since $(\gamma_1, \gamma_2) = (0, 0)$ is an equilibrium point for (34), we have $\gamma_1 = \left(\tilde{\mathbf{z}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\right) \equiv 0$, and (32) becomes

$$\tilde{\mathbf{z}} = \left(\tilde{\mathbf{z}} \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|} + \left[\tilde{\mathbf{z}} \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right)\right] \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right). \tag{36}$$

Now note that

$$\frac{d}{dt}\left(\tilde{\mathbf{z}} \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|} = -\frac{\Theta}{|\boldsymbol{\ell}|}, \tag{37}$$

and therefore

$$\tilde{\mathbf{z}}(t) \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|} = -\frac{\Theta}{|\boldsymbol{\ell}|}t + \sigma_0 \left(\frac{\mathbf{r}_0}{|\mathbf{r}_0|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|} = -\frac{\Theta}{|\boldsymbol{\ell}|}t. \tag{38}$$

We address the third term in (32) by first defining

$$\bar{\mathbf{z}} \triangleq \tilde{\mathbf{z}} - \left(\tilde{\mathbf{z}} \cdot \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|} = \left[\tilde{\mathbf{z}} \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right)\right] \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right). \tag{39}$$

Letting $\sigma \triangleq \tilde{\mathbf{z}} \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right)$, we observe that

$$\sigma(0) = \sigma_0 \left(\frac{\mathbf{r}_0}{|\mathbf{r}_0|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) \cdot \left(\frac{\mathbf{r}_0}{|\mathbf{r}_0|} \times \frac{\boldsymbol{\ell}}{|\boldsymbol{\ell}|}\right) = \sigma_0. \tag{40}$$

By direct calculation we have

$$\dot{\sigma} = -\frac{1}{2|\boldsymbol{\ell}|}a_- a_+, \tag{41}$$

and therefore

$$\sigma(t) = \sigma(0) - \frac{1}{2|\boldsymbol{\ell}|}a_- a_+ t = \frac{\sigma_0}{\rho_0}(\rho_0 + a_+ t) = -\frac{a_-}{2|\boldsymbol{\ell}|}\rho(t). \tag{42}$$

If $a_- = 0$, then the third term of (32) is identically zero and (38) yields the the second claim of our proposition. If $a_- \neq 0$, then one can verify (by way of (6) and the Jacobi identity) that

$$\dot{\bar{\mathbf{z}}} = \frac{1}{\rho}\left[a_+ \mathbb{1} - \hat{\boldsymbol{\ell}}\right] \bar{\mathbf{z}}, \tag{43}$$

and recognizing (43) as the same form as (23), we therefore have the analogous closed-form expression for $\bar{\mathbf{z}}$. The third and fourth claims of **Proposition 4** then follow from (38), (39), and (43), along with the fact that $\tilde{\mathbf{z}}(0) = \bar{\mathbf{z}}(0)$. Finally, if $\boldsymbol{\ell} = 0$, we have $\mathbf{x}_i(t) = \mathbf{x}_i^0$ (from (27)) and therefore $\dot{\tilde{\mathbf{z}}}(t) = \frac{1}{2}\left(\mathbf{x}_1^0 + \mathbf{x}_2^0\right)$, establishing the first claim of the proposition. $\square$

Remark: System behavior can be classified in terms of the initial conditions (parametrized by $\boldsymbol{\ell}$ and Θ) and the parameters a_+ and a_-. The sign and magnitude of $\boldsymbol{\ell}$ determine whether the baseline vector $\mathbf{r}$ will rotate ($\boldsymbol{\ell} \neq 0$) as well as the direction of rotation. Θ determines whether $\mathbf{r}, \mathbf{x}_1$ and $\mathbf{x}_2$ will evolve in a common plane. The parameter a_+ determines the rate of change of the inter-particle distance, and a_- determines if the center of mass will rotate. Figure 1 displays some of the possible system trajectories, including rectilinear and circling equilibria as well as an expanding spiral.

V. Relative equilibria for the n-particle case

The analysis in [4] describes the possible types of relative equilibria for an n-particle system evolving according to (1) with $SE(3)$-invariant controls. These relative equilibria correspond to

1) rectilinear formations (i.e., all particles move in the same direction with arbitrary relative positions),
2) circling formations (i.e., all particles move on circular orbits with a common radius, in planes perpendicular to a common axis),
3) helical formations (i.e., all particles follow circular helices with the same radius, pitch, axis, and axial direction of motion).

As in [4], we can express our dynamics (1) in terms of group variables $g_1, g_2, \ldots, g_n \in G = SE(3)$ as left-invariant systems

$$\dot{g}_i = g_i \xi_i, \; i = 1, 2, \ldots, n, \tag{44}$$

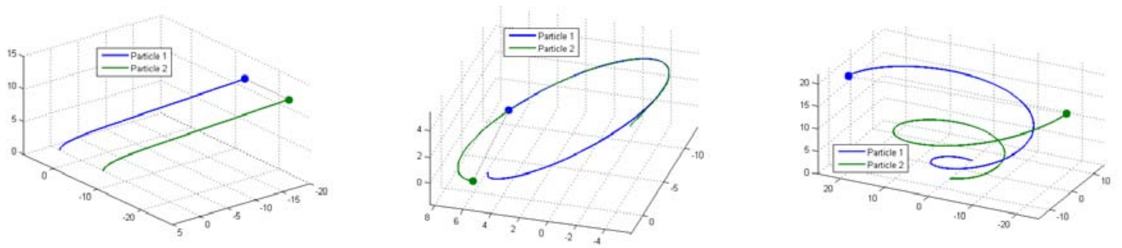
where $\xi_i \in \mathfrak{g} =$ the Lie algebra of G. Then shape variables can be defined by

$$\tilde{g}_i = g_i^{-1} g_{i+1}, \; i = 1, 2, \ldots, n, \tag{45}$$

with corresponding dynamics

$$\dot{\tilde{g}}_i = \tilde{g}_i \tilde{\xi}_i, \; i = 1, 2, \ldots, n, \tag{46}$$

where $\tilde{\xi}_i = \xi_{i+1} - \mathrm{Ad}_{\tilde{g}_i^{-1}} \xi_i \in \mathfrak{g}$. Relative equilibria for the full dynamics are equilibria for the shape dynamics (46).



(a) Rectilinear equilibrium ($\boldsymbol{\ell} = 0$) (b) Circling equilibrium ($\boldsymbol{\ell} \neq 0;\ a_- \neq 0;\ \Theta = 0;\ a_+ = 0$) (c) Expanding spiral ($\boldsymbol{\ell}, \Theta, a_-, a_+ \neq 0$)

Fig. 1. These MATLAB simulations illustrate various types of trajectories in terms of initial conditions ($\boldsymbol{\ell}$ and Θ) and parameter values (a_+ and a_-).

In our planar discussion of cyclic pursuit [2], we chose appropriate scalar variables to parametrize the shape space and aid in the analysis. However, in the current setting it is not readily apparent how best to parametrize the shape space, and this continues as a topic of ongoing research. Still, we can make the following statements concerning the existence of particular types of relative equilibria in the three-dimensional setting.

Proposition 5: Given $\{a_1, a_2, \ldots, a_n\}$, a relative equilibrium corresponding to rectilinear motion on $M_{CB(\mathbf{a})}$ under closed-loop cyclic CB pursuit dynamics (17) exists if and only if there exists a set of positive constants $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that $\sum_{i=1}^{n} \sigma_i a_i = 0$.

Proposition 6: Given $\{a_1, a_2, \ldots, a_n\}$, define $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in (0, 2\pi)$ by $(\cos\alpha_i, \sin\alpha_i) = (-a_i, \sqrt{1 - a_i^2})$. Then a *planar* circling relative equilibrium on $M_{CB(\mathbf{a})}$ under closed-loop cyclic CB pursuit dynamics (17) exists if and only if

$$\text{i. } a_i \neq \pm 1,\ i = 1, 2, \ldots, n; \quad \text{ii. } \sin\left(\sum_{i=1}^{n} \alpha_i\right) = 0. \quad (47)$$

Proof of Propositions 5 and 6: The proof for each proposition follows the same lines as the corresponding proof for the planar case in [2] and is omitted here due to space constraints. Note that the angle α_i as defined in **Proposition 6** matches the notation in the planar proof, as discussed in section III. (The choice of $\sin\alpha_i = \sqrt{1 - a_i^2}$ corresponds to CCW circling equilibria, while choosing $\sin\alpha_i = -\sqrt{1 - a_i^2}$ refers to CW circling equilibria.) Also, note that **Proposition 6** addresses the existence of circling equilibria on a common plane (rather than the more general definition of circling equilibria that permits multiple planes perpendicular to a common axis), and therefore the proof is simplified by assuming (without loss of generality) that the circling equilibrium is centered on the origin and evolves on the horizontal plane. In particular, this enables us to identify the rotation matrix $R(\alpha_i) \in SO(2)$ (from the planar proof) with the corresponding element in $SO(3)$ by the obvious inclusion map.

Remark: Observe that the constraint of **Proposition 5** is equivalent to requiring that either $a_i = 0,\ i = 1, 2, \ldots, n$ or that there exists $j, k \in [1, 2, \ldots, n]$ such that $a_j a_k < 0$. Also, observe that the condition in **Proposition 5** is not mutually exclusive with the conditions of **Proposition 6**, in contrast to the analogous planar propositions stated in [2].

VI. Conclusion

In this paper, we have extended to three dimensions the formulation and analysis of constant bearing cyclic pursuit for n interacting particles. We demonstrated the existence of an invariant manifold and then proved a result on convergence to that manifold. In the special case of $n = 2$, conservation laws give rise to integrability of the dynamics on the invariant manifold. The structure of the reduced dynamics on the invariant manifold for the setting of general n appears to also permit conserved quantities. Further details along these lines are under investigation. The results of this paper contribute to our understanding of the idea that pursuit strategies offer building blocks for collective behavior.

References

[1] M. Mischiati and P.S. Krishnaprasad, "Motion camouflage for coverage," *Proc. American Control Conference*, pp.6429-6435, 2010.
[2] K.S. Galloway, E.W. Justh and P.S. Krishnaprasad, "Geometry of cyclic pursuit," *Proc. 48th IEEE Conf. Decision and Control*, pages 7485-7490, 2009. (See also extended version at http://www.isr.umd.edu/Labs/ISL/PURSUIT).
[3] E.W. Justh and P.S. Krishnaprasad, "Steering laws for motion camouflage," *Proc. R. Soc. A*, Vol. 462, pp. 3629-3643, 2006 (see also arXiv:math.OC/0508023).
[4] E.W. Justh and P.S. Krishnaprasad, "Natural frames and interacting particles in three dimensions," *Proc. 44th IEEE Conf. Decision and Control*, 2841-2846, 2005 (see also arXiv:math.OC/0503390v1).
[5] P.V. Reddy, E.W. Justh and P.S. Krishnaprasad, "Motion camouflage in three dimensions," *Proc. 45th IEEE Conf. Decision and Control*, pp. 3327-3332, 2006 (see also arXiv:math.OC/0603176).
[6] H. K. Khalil. *Nonlinear Systems (Third Edition)*. New Jersey: Prentice-Hall, 2002.
[7] E. Wei, E.W. Justh and P.S. Krishnaprasad, "Pursuit and an evolutionary game," *Proc. R. Soc. A*, Vol. 465, pp. 1539-1559, 2009. First published online Feb. 25, 2009, doi:10.1098/rspa.2008.0480.
[8] V. A. Tucker, A. E. Tucker, K. Akers, and J. H. Enderson, "Curved flight paths and sideways vision in peregrine falcons (Falco peregrinus)," *J. Exp. Biol.*, Vol. 203, pp. 3755-3763, 2000.
[9] J. W. Lorimer, "Curved paths in raptor flight: Deterministic models," *J. Theor. Biol.*, Vol. 242, pp. 880-889, 2006.
[10] P. J. Nahin. *Chases and Escapes: The Mathematics of Pursuit and Evasion*. Princeton, NJ; Oxford, UK: Princeton University Press, 2007.