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Towards a Cell Decomposition for Rational Functions

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In this paper, we investigate a decomposition of the space of reduced rational functions of fixed degree into continued fraction cells. We give a variety of combinatorial formulas pertaining to this decomposition and investigate the effect of certain scalings on the decomposition. We conjecture that the continued fraction decomposition is indeed a cell decomposition in the topological sense. We provide evidence in low dimensions for this to be true.

1. Continued-fraction expansion

LET us denote by $g(s) = q(s)/p(s)$, a strictly proper, rational function of degree n with real coefficients, where $q(s) = q_{n-1}s^{n-1} + \cdots + q_0$ and $p(s) = s^n + p_{n-1}s^{n-1} + \cdots$. Thus $q(s)$ and $p(s)$ are coprime. We apply the Euclidean algorithm in the following manner:

$$
\begin{aligned}
p &= \alpha_1 \frac{q}{v_1} - \rho_2, \qquad && \deg \rho_2 < \deg q, \\
q &= \alpha_2 \frac{\rho_2}{v_2} - \rho_3, && \deg \rho_3 < \deg \rho_2, \\
&\vdots \\
\rho_{r-1} &= \alpha_r \frac{\rho_r}{v_r}, && \deg \rho_r = 0, \qquad \rho_r \neq 0.
\end{aligned}
\tag{1.1}
$$

The polynomials α_i are required to be monic and thus the scale factors v_i are uniquely determined. Also the coprimeness of the pair (q, p) implies that $\deg \rho_r = 0$ with $\rho_r \neq 0$. The monic polynomials α_i are called the *atoms* of (q, p). Let us define the auxiliary coefficients:

$$\beta_0 = v_1, \qquad \beta_i = v_j v_{j+1} \quad (j = 1, \ldots, r-1).$$

It is well known that the process (1.1) yields a continued fraction expansion of

q/p:

$$\frac{q}{p}=\cfrac{\beta_0}{\alpha_1-\cfrac{\beta_1}{\alpha_2-\cfrac{}{\ddots\cfrac{\beta_{r-1}}{\alpha_r}}}}. \tag{1.2}$$

We refer the reader to [13] and [8] for the relationship between the continued-fraction expansion (1.2) and the partial-realization problem. A primary result of interest to us is a classical theorem of Frobenius [5] that relates the Cauchy index of $g=q/p$ to the expansion (1.2). Recall that the Cauchy index of g is the signature $\sigma(\mathrm{H}_g)$ of the $n\times n$ Hankel matrix associated to g [7]. In our notation we can state the classical result as follows.

THEOREM (Frobenius)

$$\sigma(\mathrm{H}_g)=\sum_{i=1}^{r}\left(\operatorname{sgn}\prod_{j=0}^{i-1}\beta_j\right)\frac{1+(-1)^{\deg\alpha_i-1}}{2}. \tag{1.3}$$

For a modern proof of Frobenius' theorem using Bezoutian forms see [6].

Remark 1.1. The fraction N_r/D_r obtained after clearing all fractions in (1.2) is always irreducible. Thus, iff q and p are coprime,

$$\sum_{i=1}^{r}\deg\alpha_i=n=\deg p. \tag{1.4}$$

Remark 1.2. The generic case corresponds to $\deg\alpha_i=1$ and hence the maximum value of $r=n$.

2. Types and cells

For a continued fraction of degree n, we have an associated ordered r-tuple of integers $(\mu_1,\mu_2,\ldots,\mu_r)$ where $\mu_i=\deg\alpha_i$ and $\sum_{i=1}^{r}\mu_i=n$. Further, for fixed $(\mu_1,\ldots,\mu_r)$, it is not possible to deform continuously the real coefficients β_i in order to change their signs, without causing a drop in the degree of the continued fraction. This suggests the following definition.

DEFINITION 2.1 By the *type* of a continued fraction (1.2) we mean a pair $(\boldsymbol{\mu},\boldsymbol{\delta})$ where $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_r)$ is the r-multi-index of degrees of the atoms α_i and $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_r)$ is a string of $+1$'s and -1's obtained by setting $\delta_i=\operatorname{sgn}\beta_{i-1}$.

For the multi-index $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_r)$ we adopt the convention that $|\boldsymbol{\mu}|=\sum_{i=1}^{r}\mu_i$. For the parity string $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_r)$ we adopt the convention that $\|\boldsymbol{\delta}\|=r=$ length of the string. For fixed n, the only admissible types are pairs $(\boldsymbol{\mu},\boldsymbol{\delta})$ such that $\boldsymbol{\mu}$ is an r-multi-index where $|\boldsymbol{\mu}|=n$ and $r=\|\boldsymbol{\delta}\|$.

Now, consider the set of all continued fractions (1.2) of degree n and the same type $(\boldsymbol{\mu},\boldsymbol{\delta})$. Denote this set as $\Gamma_{\boldsymbol{\delta}}^{\boldsymbol{\mu}}$. We call $\Gamma_{\boldsymbol{\delta}}^{\boldsymbol{\mu}}$ a *continued-fraction cell* of type $(\boldsymbol{\mu},\boldsymbol{\delta})$. Each member of such a cell is uniquely specified by specifying the real numbers

$$\alpha_{k,j(k)} \quad (k=1,\ldots,r; j(k)=1,\ldots,n_k)$$

appearing in the atoms $\alpha_k(s)=s^{n_k}+\alpha_{k,1}s^{n_k-1}+\cdots+\alpha_{k,n_k}$ $(k=1,\ldots,r)$, and the real numbers $\tilde{\beta}_k$ defined by $\beta_k=\delta_k\exp\tilde{\beta}_k$ $(k=1,\ldots,r)$. Since $\sum_{k=1}^r \mu_k = n$, we conclude that the cell Γ_δ^μ is homeomorphic to the Cartesian space $\mathbb{R}^{n+r}$.

Remark 2.1. From the formula (1.3) of Frobenius, it is clear that the Cauchy index is constant over a given cell Γ_δ^μ.

3. The space rat n

Following Brockett [1], we denote as rat n the set of rational functions of the form

$$g(s)=\frac{q_{n-1}s^{n-1}+\cdots+q_0}{s^n+p_{n-1}s^{n-1}+\cdots+p_0}=\frac{q(s)}{p(s)} \tag{3.1}$$

with real coefficients such that $q(s)$ and $p(s)$ are coprime. The space rat n is visualized as an open subset of $\mathbb{R}^{2n}$ with its usual topology by identifying $g(s)$ as in (3.1) with the point $(q_{n-1},\ldots,q_0,p_{n-1},\ldots,p_0)\in\mathbb{R}^{2n}$. From [1] we have the following basic result.

THEOREM (Brockett). *The space* rat n *has* $n+1$ *arcwise connected components. The Cauchy index viewed as a continuous map from* rat n *into the set* $\{-n,-n+2,\ldots,n-2,n\}$ *distinguishes the components.* □

For $l,m\in\{0,\ldots,n\}$ with $l+m=n$, we denote as rat (l,m) the connected component

$$\{g\ :\ g\in \text{rat}\, n,\ \sigma(\mathrm{H}_g)=l-m\}.$$

The geometry of rat n has been a subject of serious study since the appearance of [1]. See also the papers [2, 3, 4, 16].

The map $g(s)\mapsto -g(s)$ is a homeomorphism of rat (l,m) onto rat (m,l). In [1], it was shown that rat $(n,0)$ is homeomorphic to $\mathbb{R}^{2n}$ and rat $(n-1,1)$ is homeomorphic to $\mathbb{R}^{2n-1}\times S^1$. Detailed information about other connected components has been elusive.

The continued fraction expansion appears to be a useful tool in this regard.

Observe that each connected component rat (l,m) has a continued-fraction decomposition, i.e. it is a union of disjoint continued-fraction cells:

$$\text{rat}\,(l,m)=\bigcup_{(\mu,\delta)\in S_{l,m}}\Gamma_\delta^\mu \tag{3.2}$$

where

$$S_{l,m}=\left\{(\mu,\delta):\ \begin{array}{l}\mu=(\mu_1,\ldots,\mu_r),\ |\mu|=l+m,\\ \delta=(\delta_1,\ldots,\delta_r),\\ \sum_{i=1}^{r}\left(\operatorname{sgn}\prod_{j=1}^{i}\delta_j\right)\frac{1+(-1)^{\mu_i-1}}{2}=l-m\end{array}\right\} \tag{3.3}$$

In the next section, we present preliminary results towards showing that the decomposition (3.2)–(3.3) of $\mathrm{rat}(l, m)$ is a cell decomposition, in a precise topological sense. Our results shed further light on the topology of $\mathrm{rat}(l, m)$. We note here that in [9, 10, 11, 12], cell decompositions based on the Kronecker and Hermite indices are presented for various spaces of system-theoretic interest.

4. Towards a topological cell decomposition of rat (l, m)

A few general definitions are needed before we proceed further.

DEFINITION 4.1 Let X be a locally compact topological space. A decomposition $\mathscr{X}_A := \{X_\alpha : \alpha \in A\}$ of X into disjoint subsets is called a *topological cell decomposition* if it satisfies:

(i) each X_α is homeomorphic to some $\mathbb{R}^{n_\alpha}$, with $n_\alpha \in \mathbb{N}$;
(ii) $\{X_\alpha : \alpha \in A\}$ is locally finite;
(iii) the boundary $\partial X_\alpha = \bar{X}_\alpha \backslash X_\alpha$ of X_α is a union of cells X_β with $\dim X_\beta < \dim X_\alpha$.

Given such a cell decomposition, we have an induced partial order on A known as the *adherence order*:

$$\alpha \leqslant \beta \quad \Leftrightarrow \quad X_\alpha \subseteq \bar{X}_\beta.$$

DEFINITION 4.2 If $q \in N$ and $\mathscr{X}_A = \{X_\alpha : \alpha \in A\}$ is a topological cell decomposition of X, then

$$\mathrm{c}_q(\mathscr{X}_A) := \mathrm{card}\,\{\alpha \in A : \dim X_\alpha = q\}$$

is called the qth *type number* of the cell decomposition $\mathscr{X}_A$. The topological cell decomposition $\mathscr{X}_A$ is said to be *finite* if A is finite (and hence all type numbers are necessarily finite).

Let X be a topological n-manifold and $\mathrm{H}_q(X, \mathbb{Z}_2)$ its qth singular homology group with coefficients in $\mathbb{Z}_2$. Let $\mathrm{b}_k(X, \mathbb{Z}_2) = \mathrm{rank}\, \mathrm{H}_k(X, \mathbb{Z}_2)$ $(k = 0, \ldots, n)$ be the mod-2 Betti numbers of X. Then the following corollary of a result in [15: p.57, Thm 3.3] appears in [12].

THEOREM *Let $\mathscr{X}_A := \{X_\alpha : \alpha \in A\}$ be a finite topological cell decomposition of a topological n-manifold X. Then*

$$\mathrm{c}_k(\mathscr{X}_A) \geqslant \mathrm{b}_{n-k}(X, \mathbb{Z}_2) \quad (k = 0, \ldots, n). \quad \square$$

In Section (4.1) below, we give explicit combinatorial formulas for determining the number of continued fraction cells of given McMillan degree and cell dimension $n + m$. In Table (4.1) we summarize our results for McMillan degree $\leqslant 6$. In Section 6, we investigate the adherence order. Our results here are only partial and, we hope, pave the way towards settling the following conjecture.

CONJECTURE *The decomposition of each connected component* $\mathrm{rat}(l, m)$ *into continued fraction cells is a cell decomposition in the topological sense.* $\square$

4.1 *Dimension formulas*

For a fixed McMillan degree we will compute now the number of cells of given dimension and Cauchy index in the decomposition arising out of the continued fraction representation.

DEFINITION 4.3 We let:

(i) $\nu^{n,k,m}$ = number of cells of McMillan degree n, Cauchy index k, and cell dimension $n+m$.

(ii) $\nu^{n,m}$ = number of cells of McMillan degree n and cell dimension $n+m$.

(iii) ν^{n} = number of cells of McMillan degree n.

Clearly, $\nu^{n,m} = \sum_{k=n}^{-n} \nu^{n,k,m}$ and $\nu^{n} = \sum_{m=1}^{n} \nu^{n,m}$.

THEOREM 4.1 *The following formulas hold.*

$$\text{(i)}\quad \nu^{n,m} = 2^m\binom{n-1}{m-1}, \qquad \text{(ii)}\quad \nu^n = 2\cdot 3^{n-1}.$$

Proof. (i) $\nu^{n,m}$ is the number of cells of the form $\Gamma^{\mu_1,\ldots,\mu_m}_{\delta_1,\ldots,\delta_m}$. Thus, it corresponds to the number of ordered partitions of the integer n into sums of m positive integers, i.e. $n=\mu_1+\cdots+\mu_m$, together with m free sign choices. The ordered partitions are best parametrized by the indices of the last elements, i.e. by $k_1<\cdots<k_m=n$, with $k_i=\sum_{j=1}^{i}\mu_j$. Since $k_m=n$, the number of such choices is $\binom{n-1}{m-1}$. The number of free sign choices is 2^m and hence (i) follows. Formula (ii) follows from the binomial expansion

$$\sum_{m=1}^{n} 2^m\binom{n-1}{m-1} = 2\sum_{m=1}^{n}\binom{n-1}{m-1}2^{m-1}1^{n-m} = 2\cdot 3^{n-1}. \quad \square$$

The computation of $\nu^{n,k,m}$ is much more complicated. We do not have an explicit formula. However, we shall provide an algorithm for this purpose.

Recall that the generic situation is given by cells of the form $\Gamma^{\mu_1,\ldots,\mu_n}_{\delta_1,\ldots,\delta_n}$, where $\mu_i=1$ and $\delta_i=\pm1$ for all i. When the parameters in the continued-fraction expansion go to infinity in certain directions, neighbouring atoms in the continued-fraction expansion coalesce into one of degree equal to the sum of the degrees.

We need to keep track of the number of coalescings in a continued fraction and one way to do this is to introduce the notion of cell configuration. By a *cell configuration,* we mean the assignment of a formal sum $\sum_{j\geq 1}\omega_j S^j$ to each continued fraction, where ω_j denotes the number of atoms of degree j in the continued fraction expansion. It is clear that this is a constant map over a cell. But, since the cell configuration does not keep track of the order of the sequence of degrees of atoms, more than one cell may have the same cell configuration. For convenience, we shall also use the phrase 'cell configuration' to mean the disjoint union of all cells that have the same formal sum assigned to them.

The generic case is assigned the cell configuration nS^1. After p coalescings we obtain cell configurations $\omega_1S^1+\cdots+\omega_{p+1}S^{p+1}$, with the following constraint, given by the McMillan degree n; that is,

$$\sum_{j=1}^{p+1} j\omega_j = n.$$

We will use $|\sum \omega_j S^j| = |\omega_1 S^1 + \cdots + \omega_{p+1} S^{p+1}|$ to denote the number of cells of this configuration. For example $|nS^1| = 2^n$. Indeed, from the generic configuration nS^1 we obtain after one coalescing the cell configuration $(n-2)S^1 + 1S^2$, i.e. $n-2$ atoms of degree one and one of degree 2. This corresponds to cells of dimension $2n-1$, one less than the generic case. The total number of cells of dimension $2n-1$ is given by $\binom{n-1}{1}2^{n-1}$, where 2^{n-1} corresponds to the number of free sign choices and $\binom{n-1}{1}$ to the position of the degree-two atom in the continued-fraction expansion. Naturally this is in agreement with Theorem 4.1. Allowing two coalescings to occur leads to two different configuration types:

$$(n-3)S^1 + S^3, \qquad (n-4)S^1 + 2S^2.$$

Now,

$$|(n-3)S^1 + S^3| = \binom{n-2}{1}2^{n-2} \quad (n \geq 3),$$

$$|(n-4)S^1 + 2S^2| = \binom{n-2}{2}2^{n-2} \quad (n \geq 4);$$

so

$$\nu^{n,n-2} = \binom{n-2}{1}2^{n-2} + \binom{n-2}{2}2^{n-2} = \binom{n-1}{2}2^{n-2} = \binom{n-1}{n-3}2^{n-2},$$

again in agreement with Theorem 4.1.

Of course, in general, a cell configuration will consist of cells of different Cauchy indices. Thus, in the generic case nS^1, the Cauchy index can be any integer in the sequence $n, n-2, \ldots, -n$. The Cauchy index $n-2k$ is obtained by exactly k negative signs in the sequence $\beta_0, \beta_0\beta_1, \ldots$. Thus

$$\nu^{n,n-2k,n} = \binom{n}{k} = \binom{n}{n-k}$$

and obviously $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. This gives us a refined description of the cells of McMillan degree n and cell dimension $2n$.

In the same way, the cell configuration $(n-2)S^1 + 1S^2$ splits according to the Cauchy index. In this case, the range of possible values of the Cauchy index is $n-2, n-4, \ldots, -(n-2)$. The Cauchy index $n-2k$ is obtained by exactly $k-1$ sign choices in the sequence $\beta_0, \beta_0\beta_1, \ldots$. Thus

$$\nu^{n,n-2k,n-1} = \binom{n-1}{1}2\binom{n-2}{k-1} = 2(n-1)\binom{n-2}{k-1}.$$

Here, the factor 2 corresponds to the two possible signs of the degree-two atom, $\binom{n-1}{1}$ corresponds to its position in the continued-fraction expansion, and $\binom{n-2}{k-1}$ corresponds to the $k-1$ sign choices of the degree-one atoms. It is easy to check that

$$\sum_{k=1}^{n-1} 2\binom{n-1}{1}\binom{n-2}{k-1} = 2(n-1)2^{n-2} = 2^{n-1}\binom{n-1}{n-2}.$$

so that indeed

$$\sum_{k=1}^{n-1} \nu^{n,n-2k,n-1} = \nu^{n,n-1}.$$

Cells of dimension $2n-2$ correspond to two coalescings, and hence the corresponding cell configurations are $(n-3)S^1+S^3$ and $(n-4)S^1+2S^2$. In the cell configuration $(n-3)S^1+S^3$, the range of values of the Cauchy index is $n-2, n-4, \ldots, -(n-2)$. The Cauchy index $n-2k$ is obtained by $k-1$ sign choices so we have $\binom{n-2}{1}\binom{n-2}{k-1}$ cells giving Cauchy index $2n-k$. In the cell configuration $(n-4)S^1+2S^2$, the range of values of the Cauchy index is $n-4, \ldots, -(n-4)$. Here we assume implicitly $n \geqslant 4$. The number of cells in this configuration yielding Cauchy index $n-2k$ is clearly $2^2\binom{n-2}{2}\binom{n-4}{k-2}$. For $k \geqslant 2$, the number of cells with cell dimension $2n-2$ and Cauchy index $n-2k$ is

$$\nu^{n,n-2k,n-2} = \binom{n-2}{1}\binom{n-2}{k-1} + 2^2\binom{n-2}{2}\binom{n-k}{k-2}.$$

Now

$$\begin{aligned}\nu^{n,n-2} &= \sum_{k=1}^{n-1}\binom{n-2}{1}\binom{n-2}{k-1} + \sum_{k=2}^{n-2} 2^2\binom{n-2}{2}\binom{n-4}{k-2} \\ &= \binom{n-2}{1}2^{n-2} + 2^2\binom{n-2}{2}\cdot 2^{n-4} = 2^{n-2}\left[\binom{n-2}{1}+\binom{n-2}{2}\right] \\ &= 2^{n-2}\binom{n-1}{2},\end{aligned}$$

again in accordance with Theorem 4.1.

Now we present the general arguments. Given any configuration $\sum_1^{p+1} \omega_j S^j$ obtained after p coalescings, the two constraints

$$\sum_{j=1}^{p+1} j\omega_j = n, \qquad \sum_{j=1}^{p+1} \omega_j = n-p \tag{4.1a,b}$$

must be satisfied. Let $\Omega_{n,p}$ be the set of all vectors $(\omega_1, \ldots, \omega_{p+1})$ that are nonnegative integral solutions of the equations (4.1). Equation (4.1b) implies that there are exactly 2^{n-p} sign choices in this cell configuration. The number of orderings is given by the multinomial formula

$$\sum_{\omega\in\Omega_{n,p}} \frac{(n-p)!}{\omega_1!\cdots\omega_{p+1}!}.$$

Thus

$$\left|\sum_{\omega\in\Omega_{n,p}} \omega_j S^j\right| = 2^{n-p} \sum_{\omega\in\Omega_{n,p}} \frac{(n-p)!}{\omega_1!\cdots\omega_{p+1}!}.$$

Comparing with Theorem 4.1(i), we conclude that under these constraints the following combinatorial formula holds:

$$\binom{n-1}{n-p-1} = \sum_{\omega\in\Omega_{n,p}} \frac{(n-p)!}{\omega_1!\cdots\omega_{p+1}!}.$$

(It might be of interest to give a direct proof of this formula and have some intuitive interpretation of it.)

It is instructive to compute the numbers $\nu^{n,n-2k,n-p}$ by the use of equations (4.1). We observed that

$$\left|\sum_{\omega\in\Omega_{n,p}} \omega_j S^j\right| = 2^{n-p} \sum_{\omega\in\Omega_{n,p}} \frac{(n-p)!}{\omega_1!\cdots\omega_{p+1}!}.$$

However, the cells in the cell configuration $\sum \omega_j S^j$ are distributed in different connnected components of rat n corresponding to different Cauchy indices. In fact, after p coalescings the Cauchy index range is from $n-2\lfloor\frac{1}{2}(p+1)\rfloor$ to $2\lfloor\frac{1}{2}(p+1)\rfloor - n$. The number of cells in the cell configuration $\sum \omega_j S^j$ that have Cauchy index $n-2k$ will be denoted by $|\sum \omega_j S^j|_k$.

If we denote by σ_+ and σ_- the sum of the even ω_i and odd ω_j of the cell configuration, i.e.

$$\sigma_+ = \sum_{j=1}^{\lfloor\frac{1}{2}(p+1)\rfloor} \omega_{2j}, \qquad \sigma_- = \sum_{j=0}^{\lfloor\frac{1}{2}p\rfloor} \omega_{2j+1},$$

then we have the following result.

THEOREM 4.2

$$\left|\sum_{j=1}^{p+1} \omega_j S^j\right|_k = 2^{\sigma_+} \frac{(n-p)!}{\omega_1!\cdots\omega_{p+1}!} \binom{\sigma_-}{k-\frac{1}{2}(n-\sigma_-)}. \quad \square$$

Next we work out some specific examples. If $p=3$, then the solution vectors to the equation pair

$$\omega_1+\omega_2+\omega_3+\omega_4 = n-3, \quad \omega_2+2\omega_3+3\omega_4 = 3$$

are $(n-4,0,0,1)$, $(n-4,1,1,0)$, and $(n-6,3,0,0)$, and the corresponding cell configurations are

$$(n-4)S^1+S^4, \qquad (n-5)S^1+S^2+S^3, \qquad (n-6)S^1+3S^2$$

respectively. Now,

$$|(n-4)S^1+S^4|_k=2\binom{n-3}{1}\binom{n-4}{k-2}\quad(k\geqslant 2),$$

$$|(n-5)S^1+S^2+S^3|_k=2\frac{(n-3)!}{(n-5)!\,1!\,1!}\binom{n-4}{k-2}\quad(k\geqslant 2),$$

$$|(n-6)S^1+3S^2|_k=2^3\binom{n-3}{3}\binom{n-6}{k-3}\quad(k\geqslant 3).$$

For $p=4$, the solution vectors to the equation pair

$$\omega_1+\cdots+\omega_5=n-4,\quad \omega_2+2\omega_3+3\omega_4+4\omega_5=4$$

are

$$(n-5,0,0,0,1),\quad (n-6,0,2,0,0),\quad (n-6,1,0,1,0),$$
$$(n-7,2,1,0,0),\quad (n-8,4,0,0,0),$$

and the corresponding cell configurations are

$$(n-5)S^1+S^5,\qquad (n-6)S^1+2S^3,\qquad (n-6)S^1+S^2+S^4,$$
$$(n-7)S^1+2S^2+S^3,\qquad (n-8)S^1+4S^2,$$

respectively. A simple computation leads to

$$|(n-5)S^1+S^5|_k=\binom{n-4}{1}\binom{n-4}{k-2}\quad(k\geqslant 2),$$

$$|(n-6)S^1+2S^3|_k=\binom{n-4}{2}\binom{n-4}{k-2}\quad(k\geqslant 2),$$

$$|(n-6)S^1+S^2+S^4|_k=2^2\frac{(n-4)!}{(n-6)!\,1!\,1!}\binom{n-6}{k-3}\quad(k\geqslant 3),$$

$$|(n-7)S^1+2S^2+S^3|_k=2^2\frac{(n-4)!}{(n-7)!\,2!\,1!}\binom{n-6}{k-3}\quad(k\geqslant 3),$$

$$|(n-8)S^1+4S^2|_k=2^4\binom{n-4}{4}\binom{n-8}{k-4}\quad(k\geqslant 4).$$

It is of interest to compute the alternating sums $\sum_{p=0}^{n-1}(-1)^p v^{n,n-2k,n-p}$. We cannot yet give a general formula for this alternating sum. However we compute this for small values of k. Indeed, since after p coalescings the Cauchy index is bounded by $n-2\lfloor\frac{1}{2}(p+1)\rfloor$, only the values $p=0,1,2$ contribute to the Cauchy index $n-2$ and only the values $p=0,1,2,3,4$ contribute to the Cauchy index $n-4$. Now

$$\sum_{p=0}^{2}(-1)^p v^{n,n-2,n-p}=\binom{n}{1}-2\binom{n-2}{0}\binom{n-1}{1}+\binom{n-2}{0}\binom{n-2}{1}=0.$$

Similarly, if $k = 2$,

$$\sum_{p=0}^{4} (-1)^p v^{n,n-4,n-p} = \binom{n}{2} - 2\binom{n-2}{1}\binom{n-1}{1} +$$

$$\binom{n-2}{1}\binom{n-2}{1} + \binom{n-4}{0}\binom{n-2}{2}2^2 - 2\binom{n-3}{1}\binom{n-4}{0} - 2\frac{(n-3)!}{(n-5)!}\binom{n-4}{0}$$

$$+\binom{n-4}{1}\binom{n-4}{0} + \binom{n-4}{2}\binom{n-4}{0} = 0$$

as can be verified by expanding terms. These examples as well as others from Table 4.1 lead us to conjecture that for all $1 \leq k \leq n-1$ we have

$$\sum_{p=0}^{n-1} (-1)^p v^{n,n-2k,n-p} = 0. \tag{4.2}$$

Remark 4.1. The specific examples worked out above suggest the following algorithm for computing $v^{n,n-2k,n-p}$. First determine the set $\Omega_{n,p}$. Then for the given k and each member ω of this set, use the formula in Theorem 4.2 and add up the results to obtain the required number $v^{n,n-2k,n-p}$.

Remark 4.2. The expression on the left of equation (4.2) is an Euler-characteristic-like expression.

TABLE 4.1

n	$n+m$	Cauchy index (k)													$v^{n,m}$
		6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	
1	2						1		1						2
2	4					1		2		1					4
2	3							2							2
3	6				1		3		3		1				8
3	5						4		4						8
3	4						1		1						2
4	8			1		4		6		4		1			16
4	7					6		12		6					24
4	6					2		8		2					12
4	5							2							2
5	10		1		5		10		10		5		1		32
5	9				8		24		24		8				64
5	8				3		21		21		3				48
5	7						8		8						16
5	6						1		1						2
6	12	1		6		15		20		15		6		1	64
6	11			10		40		60		40		10			160
6	10			4		40		72		40		4			160
6	9					18		44		18					80
6	8					3		14		3					20
6	7							2							2

Entries in columns 3 to 15 are values of $v^{n,k,m}$

5. Scaling and continued fraction cells

In [2], the authors present an analysis of the action of certain natural scaling groups on the space rat n. Consider the one-parameter groups:

$$\begin{aligned}
&(1)\ s \to \alpha s,\ \alpha \in \mathbb{R}_+ \quad \text{(frequency scaling);}\\
&(2)\ s \to s + \sigma,\ \sigma \in \mathbb{R} \quad \text{(shift of origin);}\\
&(3)\ g(s) \to mg(s),\ m \in \mathbb{R}_+ \quad \text{(magnitude scaling);}\\
&(4)\ g(s) \to g(s)/[1 + kg(s)],\ k \in \mathbb{R} \quad \text{(feedback);}\\
&(5)\ g(s) = \boldsymbol{c}^{\mathrm{T}}(sI - A)^{-1}\boldsymbol{b} \to \boldsymbol{c}^{\mathrm{T}}(sI - A)^{-1}\mathrm{e}^{A\tau}\boldsymbol{b},\ \tau \in \mathbb{R} \quad \text{(time shift).}
\end{aligned} \tag{5.1}$$

The scalings are collected into sets A and B as $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 5\}$. The scalings in set A and set B generate, respectively, two four-parameter Lie groups G_A and G_B that act on rat n. The action of G_A is defined by

$$\phi_A : G_A \times \text{rat}\, n \to \text{rat}\, n : ((\alpha, m, \sigma, k), g(s)) \mapsto \frac{mg(\alpha s + \sigma)}{1 + kmg(\alpha s + \sigma)}. \tag{5.2}$$

The action of G_B is defined by

$$\begin{aligned}
&\phi_B : G_B \times \text{rat}\, n \to \text{rat}\, n\\
&\quad : ((m, \sigma, \alpha, \tau), g(s) = \boldsymbol{c}^{\mathrm{T}}(sI - A)^{-1}\boldsymbol{b}) \mapsto m\boldsymbol{c}^{\mathrm{T}}[(\alpha s - \sigma)I - A]^{-1}\mathrm{e}^{A\tau}\boldsymbol{b}.
\end{aligned} \tag{5.3}$$

From [2] it is known that the scalings (5.1) and hence the actions (5.2)–(5.3) leave invariant a connected component rat (l, m). We are interested in how the scalings respect the continued-fraction expansion (1.2). We leave it to the reader to verify the results of Table 5.1.

From Table 5.1 and the definition of a continued fraction cell in Section 2 we conclude the following result.

THEOREM 5.1 *The action ϕ_A of G_A leaves invariant the continued fraction cell decomposition.* □

The time-shift scaling (5) does not leave invariant the cell decomposition. This can be seen by the following example.

EXAMPLE 5.1 Let $a > 0$. Then $g(s) = (as + 1)/s^2$ belongs to rat (1, 1). Under the time shift τ (scaling (5)), $g(s)$ transforms into $g_\tau(s) = [(\tau + a)s + 1]/s^2$. Now $g(s)$

TABLE 5.1

Scaling	Invariants of continued fraction
1	degrees of atoms; signs of β_i.
2	degrees of atoms; values of β_i.
3	all atoms; all β_i except β_0.
4	all atoms except α_1 which is perturbed by a constant; all β_i.

has the continued fraction expansion

$$\frac{a}{(s-1/a)+\dfrac{1/a^2}{(s+1/a)}} \tag{5.4}$$

and hence $g(s)$ belongs to the cell $\Gamma^{1,1}_{1,-1}$. But, for $\tau=-a$, we have $g_\tau(s)=1/s^2 \in \Gamma^2_1$.

One thus expects the action ϕ_B to mix up the cells. Details of this process will be explored in a forthcoming paper.

We note that in [7] it was shown that the groups G_A and G_B act freely on rat (l, m) iff $|l-m|>1$. However, for G_A to act freely on a cell and at the same time leave invariant the cell it is necessary that the cell be of dimension ≥ 4. From Section (4.1), we see that this is true for all cells if $n \geq 3$. Thus the condition that the magnitude of the Cauchy index exceeds unity gives us more refined information about the action of the scaling group G_A when restricted to a cell.

Before we close this section we would like to draw attention to a different decomposition of rat (l, m) presented in [14]. The main result of that paper is as follows.

THEOREM [Krishnaprasad]

Each connected component of rat (l, m), with $l+m=n$, admits an n-dimensional foliation whose leaves are diffeomorphic to $T^k \times \mathbb{R}^{n-k}$, where k is constant on an open set. Further, on rat $(n, 0)$, $k=0$ and the foliation is a trivial fibration. Here T^k denotes the k-torus and the largest value k can take on a connected component of rat (l, m) is

$$k_{\max} = \lfloor \tfrac{1}{2}(n-|l-m|) \rfloor. \quad \square$$

The integrable n-plane distribution associated to the foliation in the above theorem is generated by n vector fields $X^0, X^1, \ldots X^{n-1}$, defined by

$$X^r = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ([A^c(p)]^r)_{i+1,j+1} q_j \frac{\partial}{\partial q_i},$$

where $r=0, \ldots, n-1$, and $A^c(p)$ denotes the unique companion form matrix associated to the polynomial $p(\lambda)=\lambda^n+p_{n-1}\lambda^{n-1}+\cdots+p_0$.

Note that the vector fields X^0 and X^1 generate respectively the magnitude (3) and shift (5) scalings. The flows of the vector fields X^r leave invariant the poles of the rational functions and hence we conclude that all rational functions on the same leaf have the same poles. It would be interesting to understand the geometrical relationships between the foliation above and the decomposition of rat (l, m) into continued-fraction cells.

6. On the adherence order

In order for the continued-fraction decomposition to be a cell decomposition in the topological sense, it is essential that the cells fulfil the closure condition, i.e., the boundary of a cell consists of cells of lower dimension. We do not at present have methods for verifying that this holds for an arbitrary cell in an arbitrary connected component rat (l, m). However, calculations in low dimensions appear to confirm this.

We will use the notation

$$\Gamma^{\alpha_1,\ldots,\alpha_m}_{\sigma_0,\ldots,\sigma_{m-1}} \to \Gamma^{\alpha'_1,\ldots,\alpha'_{m-1}}_{\sigma'_0,\ldots,\sigma'_{m-2}}$$

for a permissible cell coalescing. We will also call it a *transition rule.* A cell is called a *terminal cell* if no more coalescings are possible. Naturally any cell of the form Γ^{α}_{β} is terminal. So are all cells of the form

$$\Gamma^{\alpha_1,\ldots,\alpha_k}_{\sigma_0,\ldots,\sigma_{k-1}}$$

with all α_i odd and all σ_i equal to 1, or those with all α_i odd and all σ_i equal to 1 except σ_0 equal to -1. This is clear from the constraint that the Cauchy index remains invariant under coalescing.

To verify that the appropriate transition rules are realizable, we have taken a brute-force approach in low dimensions (McMillan degree <4). In this the shift scaling has been found to be useful since it does *not* preserve the cell decomposition. We plan to present details in a future paper.

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References

Brockett, R. W. 1976 Some geometric questions in the theory of linear systems, *IEEE Trans. autom. Control* AC-**21,** 449–455.

Brockett, R. W. & Krishnaprasad, P. S. 1980 A scaling theory for linear systems. *IEEE Trans. autom. Control.* AC-**25,** 197–207.

Byrnes, C. I. & Duncan, T. E. 1981 On certain topological invariants arising in system theory. Pp. 29–71 in *New Directions in Applied Mathematics* (Ed. G. Young). Springer, New York.

Delchamps, D. F. 1985 Global structure of families of multivariable linear systems with an application to identification. *Math. Syst. Theor.* **18,** 329–380.

Frobenius, F. G. 1895 Ueber das Traegheits gesetz quadratischer Formen. *J. reine angew. Math.* **114,** 187–230.

Fuhrmann, P. A. 1983 On symmetric rational transfer functions. *Lin. Alg. & Applics* **50,** 167–250.

Gantmacher, F. R. 1959 *The Theory of Matrices.* Chelsea, New York.

Gragg, W. & Lindquist, A. 1983 On the partial realization problem. *Lin. Alg. & Applics* **50,** 277–319.

Helmke, U. 1982 *Zur Topologie des Raumes linearer Kintrollsysteme.* Ph.D. Thesis, University of Bremen.

Helmke, U. 1982 The topology of the space of linear systems. *Proc. 21st IEEE Conf. Decision and Control,* pp. 948–949. IEEE.

Helmke, U. 1983 A cell decomposition for spaces of linear systems. Development et

Utilisation d'Outils et Modeles Mathematiques en Automatique, Analyse de Systemes et Traitement de Signal, *Coll. CNRS, RCP* 567, Belle-Ile, pp. 623–637.

HELMKE, U., & HINRICHSEN, D. 1983 Canonical forms and orbit spaces of linear systems. Forschungsschwerpunkt Dynamische Systeme, Report no. 97, Univesität Bremen.

KALMAN, R. E. 1979 On partial realizations, transfer functions and canonical forms. *Acta Polytechnica Scandinavica* Ma **31,** 9–32.

KRIKSHNAPRADAD, P. S. 1979 Symplectic mechanics and rational functions. *Richerche di Automatica* **10,** 107–135.

MASSEY, W. 1978 *Homology and Cohomology Theory*. Marcel-Dekker, New York.

SEGAL, G. 1979 The topology of spaces of rational functions. *Acta Mathematica* **143,** 39–72.