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Geometric Methods for Multibody Dynamics

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Abstract: In this paper, we use differential geometric methods to understand the dynamics and control of certain multibody systems. Specifically, we treat rigid spacecraft with rotors and announce a basic stability theorem for the dual-spin maneuver. We then show how to decouple the effect of disturbance torques from spacecraft attitude variables.

Our stability arguments based on Lie-Poisson structures involve general principles that are applicable to more complex multibody systems than those considered here. The techniques for decoupling disturbance torques that we use here also admit applications to more complex multibody systems including elastic elements.

1. Introduction

There has been a tremendous resurgence of interest in the subject of analytical mechanics in recent years. this has been partly due to the systematic infusion of a rich variety of geometric ideas and techniques into the foundations of the subject. Recent discoveries of new inegrable classes of systems (both finite and infinite dimensional), phenomena related to 'chaos' and recent developments in stability and critical point theory of mechanical systems have relied heavily on geometric and algebraic ideas. Along side these developments, there has been a steady effort on the part of control theorists to understand problems of control and estimation of nonlinear systems using geometric tools.

In this paper, we aim to show certain geometric methods and ideas in action, - in solving concrete problems related to attitude control of spacecraft. We focus on the simplest class of multibody spacecraft, namely rigid spacecraft carrying symmetric rotors. There is a Hamiltonian structure underlying this class of dynamical systems even when the rotors are driven. A complete understanding of this fact would necessitate an excursion into the recent developments in Hamiltonian systems with inputs and outputs due to Brockett, Takens, Willems and Van-der-Schaft (see [1], [16], [19], [17]). Instead we outline a treatment based on Lie-Poisson structures and announce a basic stability theorem for dual-spin spacecraft. Details of the proof are to be found in [9], [10].

We then proceed to solve the problem of decoupling a spacecraft with momentum wheels from internal disturbance torques. This is achieved by designing appropriate nonlinear feedback laws for the driving torques on the wheels. To keep the paper self-contained an exposition of the abstract disturbance decoupling theory is also given.

2. Dynamics of Rigid Spacecraft with Rotors

The equations of motion for a rigid spacecraft carrying multiple symmetric rotors may be obtained by systematic application of Newton's laws. In the notation of Wittenburg [20], these are,

$$J\dot{\omega} + \sum_{i=1}^{m} \overset{\circ}{h}_i + \omega \times \left[J\cdot\omega + \sum_{i=1}^{m} h_i + \sum_{i=m+1}^{n} h_i \right]$$

$$= M - \sum_{i=m+1}^{n} h_i \ , \qquad (2.1a)$$

$$u_i \cdot (J_i^r \dot{\omega} + \overset{\circ}{h}_i) = M_i^r, \ i=1,2,\ldots m, \qquad (2.1b)$$

where, we assume that of a total of (m+n) rotors each indexed by i the first m are subject to known axial torque components M_i^r and the remaining n have known angular momenta $h_i(t)$ with respect to the body of the spacecraft. In the scalar equations (2.1.b), u_i denotes a unit vector along the axis of the i^{th} rotor, and (o) denotes differentiation with respect to a spacecraft frame.

J_i^r = moment of inertia of i^{th} rotor about its spin axis and J = moment of inertia of spacecraft with all rotors locked. ω is the spacecraft body angular velocity and h_i is the angular momentum of the i^{th} rotor relative to the spacecraft. M denotes the resultant of external torques.

Suppose now that one is interested in understanding the dual-spin maneuver. Here the intuition is the following : if the spacecraft contains a driven rotor spinning at a sufficiently high constant relative angular velocity, then in the presence of a suitable additional damping mechanism, the spacecraft body angular velocity eventually converges to an (unique) equilibrium spin. Although many attempts have been made in analyzing the dual-spin turn, our stability theorem below apears to be the first rigorous verification of the above intuition for a model linear damping mechanism based on rotors.

We assume that the spacecraft has two sets of three symmetric rotors each; one set free-spinning with linear damping and one set driven at constant relative angular velocities. The equations of motion (2.1.a-b) now take the form,

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$$\dot{h}_v = S(J^{-1}h_v)\,[h_v + h_d + h_w] - \gamma h_v + \delta h_d$$

$$\dot{h}_d = \gamma h_v - \delta h_d \qquad (2.2)$$

$$\dot{h}_w = 0,$$

where, $h_d = J_a^d\,(\omega_a + \omega_a^d)$ is the 3-vector of angular momenta of the damping rotors with respect to inertial space; $h_w = J_a^w \omega_a^w$ is the 3-vector of angular momenta of the driven rotors relative to the spacecraft; $h_v = J\omega_a$ and ω_a = spacecraft angular velocity vector ; J_a^v = moment of inertia of spacecraft with all rotors locked and

$J = J_a^v - J_a^d$; $\gamma = \alpha J^{-1}$, $\delta = \alpha J_a^{d-1}$ and α = diagonal matrix of positive damping coefficients of the free-spinning wheels. Among the main results in [9], [10] we mention:

(1) as $\alpha \to 0$, the system (2.2) tends to a Hamiltonian system;

(2) the Hamiltonian structure in (1) is not canonical but is a Lie-Poisson structure;

(3) if $\alpha > 0$, then for h_w sufficiently large, we have proved an asymptotic stability theorem.

The details are to be found in [9]. We enlarge upon items 1 and 2 above.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with braqcket $[\cdot,\cdot]$. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. The space $F(\mathfrak{g}^*)$ of smooth real-valued functions on $\mathfrak{g}^*$ carries a Poisson structure (bracket) as follows:

$$\{\cdot,\cdot\} : F(\mathfrak{g}^*) \times F(\mathfrak{g}^*) \to F(\mathfrak{g}^*)$$

$$(\phi\ ,\ \psi) \to \{\phi,\psi\}$$

$$\{\phi,\ \psi\}\ (f) = \langle f,\ [\ \frac{\delta\phi}{\delta f}\ ,\ \frac{\delta\psi}{\delta f}\ ]\rangle,$$

Where $f \in \mathfrak{g}^*$, $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and $\frac{\delta\phi}{\delta f}$, $\frac{\delta\psi}{\delta f}$ denote Lie algebra gradients ([4], [5], [11], [12], [18]). Given H (a Hamiltonian) in $F(\mathfrak{g}^*)$, the vector field X_H is defined by setting

$$X_H(\phi) = \{H,\phi\}\ .$$

If x denotes a global coordinate system on $\mathfrak{g}^*$ then the differential equation,

$$\frac{dx}{dt} = X_H(x)$$

is known as the Lie-Poisson equation associated to $\mathfrak{g}^*$, H and the chosen basis. The vector fields X_H leave invariant the coadjoint orbits in $\mathfrak{g}^*$ (see [9]).

In [9] we showed that the system (2.2) with $\alpha = 0$ (and hence $\gamma = \delta = 0$) is in Lie-Poisson form, with

$$\mathfrak{g} = so(3) \oplus R^3 \oplus R^3$$

$$H = \frac{1}{2} \langle h_v,\ J^{-1} h_v\rangle + \frac{1}{2} \langle h_d,\ J_a^{d-1}\ h_d\rangle$$

The coadjoint orbits are spheres and are invariant ($\Leftrightarrow$ total body angular momentum is conserved). The Hamiltonian H = T-Q-W, where T = total kinetic energy of spacecraft, Q = total energy supplied through driven rotors, W = kinetic energy of driven rotors (maintained constant).

If $\alpha > 0$, we showed in [9] that for a range of values of h_w (given by the perfectness conditions of [10]), the system (2.2) is asymptotically stable in the large (i.e all trajectories converge to one of the equilibrium points). The Lyapunov function used to establish this is V = H+W. The key observation here is that W is an element in the center the Poisson bracket algebra $F(\mathfrak{g}^*)$. The addition of such an element (called a Casimir element) to the Hamiltonian gives us a Lyapunov function. This is not accidental and is part of a rather general picture see ([4], [11]).

We have

<u>Theorem</u>: Assume that h_w is large enough for the perfectness conditions of ([9] [10]) to hold. Then almost all trajectories of (2.2) converge to the unique global minimum of V on the momentum variety,

$$\| h_v + h_d + h_w \|^2 = \mu^2 = \text{constant}$$

3. <u>Disturbance Decoupling in Spacecraft</u>

Spacecraft control system designers must design attitude control systems that shield or decouple the attitude variables from a variety of disturbance effects. These include,

(a) external forces and torques due to gravity gradient, solar pressure on panels, aerodynamic drag at low altitudes, magnetic field interactions etc.;

(b) internal forces and torques due to crew motion, internal reconfiguration (e.g. space shuttle manipulator motions), fuel sloshing etc.

When accurate models of such disturbances we available, it is possible to design specific compensation schemes. For a recent survey of the literature and an overview of techniques, see [15].

In this paper, we show how to design nonlinear feedback control laws for momentum wheels which can decouple a part of the attitude dynamics (more precisely, a row of the direction cosine matrix) from internal disturbance torques. This is a long exercise in Lie bracket calculations and solution of first order partial differential equaions. The only previous effort of this nature (that we are aware of) is in [14]. In that paper the authors treat the (simpler) problem of decoupling (from external disturbances), rigid body dynamics using reaction jets.

Eliminating the rotor dynamics from (2.1) and relabeling certain variables we get the following basic model,

$$\dot{A} = S(\omega)A$$

$$\dot{\omega} = J^{-1}S(\omega)Ah + g_1 u_1 + g_2 u_2 + pw$$

where A = direction cosine matrix, ω = body angular velocity vector, $g_1 = (1,0,0)'$, $g_2 = (0,1,0)'$ and $p = (0,0,1)$; $J = J_a^v - J_a^w$, h = conserved angular momentum vector.

The normalized momentum wheel torques are denoted as $u_1(t)$ and $u_2(t)$ and $w(t)$ is an (internal) disturbance torque.

Our aim is to design a feedback law, of the form $u_1 = f_1(\omega,A)$ and $u_2 = f_2(\omega,A)$ such that the last row of the attitude matrix A is unaffected by the disturbance $\omega(\cdot)$.

In section 4 below, we outline the general geometric framework for disturbance decoupling. The basic ideas appeared in linear system theory during the period 1969-1975 and a comprehensive exposition may be found in [22] (see also [21]). Nonlinear decoupling methods are of recent origin and the basic results are in [3], [6].

Our solution to the disturbance decoupling problem for spacecraft using momentum wheels is given in seciton 5. Since repeated use of differential geometric notation is made in the following pages we note here :

(i) given f(x) and g(x),two smooth n-vector functions viewed as local coordinate representation of two vectorfields in R^n, the Lie bracket $[f,g]$ has the local coordinate representation

$$[f,g] := (\frac{\partial g}{\partial x})f - (\frac{\partial f}{\partial x})g$$

(ii) The Lie derivative of a smooth function $\alpha(x)$ along a vector field $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ is given by

$$X\alpha = \sum_{i=1}^{n} f_i \frac{\partial \alpha}{\partial x_i}$$

4. Disturbance Decoupling (an outline):

An analytic nonlinear control system on a manifold M (= phase space) may be represented in local coordinates in the form

$$\Sigma : \quad \begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \ , \end{aligned} \qquad 4.1$$

where $G(x) = [g_1(x), \cdots, g_m(x)]$, f, g_i are analytic vector fields and h is an analytic output map.

By a (smooth) distribution Δ on M we mean a (smooth) choice of a subspace $\Delta_x \subset TM_x$ of the tangent space at each point $x \in M$.

A distribution Δ is invariant under the dynamics Σ if

$$\begin{aligned} &[f, \Delta] \subset \Delta \\ &[g_i, \Delta] \subset \Delta \ . \quad i=1,2,\cdots, m \end{aligned} \qquad 4.2$$

Have $[\cdot,\cdot]$ denotes Lie bracketting.

The involutive closure $\bar{\Delta}$ of a distribution Δ is the smallest distribution $\bar{\Delta}$ containing Δ and satisfying,

$$[\bar{\Delta}, \bar{\Delta}] \subset \bar{\Delta} \qquad 4.3$$

Fact: If Δ is invariant under Σ then so is $\bar{\Delta}$.

Let Δ be an involutive (i.e. $\bar{\Delta} = \Delta$), invariant distribution of constant rank k for the system Σ. We obtain a reduction theorem for Σ as follows:

Let (x_1, x_2) be local coordinates such that $x_1 \epsilon R^{n-k}$ and $x_2 \epsilon R^k$ respectively.

and

$$\Delta_x = \text{span} \{ \frac{\partial}{\partial x_2} \}$$

In these coordinates the system Σ becomes

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + G_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_1, x_2) + G_2(x_1, x_2)u \\ y &= h(x_1, x_2). \end{aligned} \qquad 4.4$$

But since Δ is invariant,

$$[f, \frac{\partial}{\partial x_2}] \ \epsilon \ \text{span} \{ \frac{\partial}{\partial x_2} \} \ .$$

This implies $\frac{\partial f_1}{\partial x_2} = 0$

Similarly $\frac{\partial G_1}{\partial x_2} = 0$

Thus f_1, G_1 are functions of x_1 alone and we obtain

$$\Sigma_p : \dot{x}_1 = f_1(x_1) + G_1(x_1)u \qquad 4.5$$

a subsystem of Σ.

For the purposes of disturbance decoupling we need a modified notion of invariance.

Let $\alpha(x) \underline{\Delta} (\alpha_1(x), \ldots, \alpha_m(x))'$ denote a smooth R^m-valued function of x and let $\beta(x) \underline{\Delta} [\beta_{ij}(x)]_{m \times m}$ denote a smooth mxm matrix valued (invertible) function of x. We interpret $\alpha(x)$ as defining a nonlinear feedback and $\beta(x)$ as defining a change of coordinates in the input space which depends nonlinearly on x.

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Let $(G\beta)_i$ denote the i^{th} column of the matrix $G\beta$ and let $f+G\alpha$ denote the closed loop drift vectorfield.

We say that a distribution Δ is (f, G) <u>invariant</u> if there exist $\alpha(x)$ and $\beta(x)$ such that

$$[f + G\alpha, \Delta] \subset \Delta$$
$$[(G\beta)_i, \Delta] \subset \Delta \tag{4.6}$$

In other words there is a nonlinear feedback law α and a nonlinear change of coordinates β in input space such that Δ is invariant under the new dynamics

$$\tilde{f} = f + G\alpha$$
$$\tilde{G} = G\beta$$

Often it is difficult to establish (f,G) invariance. But there is a related concept:

A distribution Δ is <u>locally (f,G) invariant</u> if

$$[f, \Delta](x) \subset \Delta(x) + \text{span}\{g_i(x)\}_{i=1}^m$$
$$[g_i, \Delta](x) \subset \Delta(x) + \text{span}\{g_i(x)\}_{i=1}^m \quad .$$

<u>Lemma</u>: [7] Suppose Δ is <u>locally (f,G) invariant</u> and $\bar{\Delta}$ is its involutive closure and the dimensions of $\bar{\Delta}(x)$, span $\{g_i(x)\}_{i=1}^m$ and $\bar{\Delta}(x) \cap \text{Span}\{g_i(x)\}_{i=1}^m$ are constant. Then locally around each $x \in M$ there exists an $\alpha(x)$ and an invertible $\beta(x)$ satisfying (4.6).

The concept of (f,G) invariance plays a crucial role in the disturbance decoupling problem.

We outline the main ideas behind disturbance decoupling. Consider a system

$$\Sigma_d : \quad \begin{aligned} \dot{x} &= f(x) + G(x)u + P(x)w \\ y &= h(x) \end{aligned}$$

Here the vector $u \in R^m$ is a control and the vector $w \in R^p$ is a (time-dependent) disturbance. One says that the system Σ_d is <u>disturbance decoupled</u> if the output y is independent of the disturbance w. Now a given system Σ_d may not have this property. So one can try to modify the system using feedback $\alpha(x)$ and input change of coordinates $\beta(x)$ such that the modified system is disturbance decoupled. This is the essential idea. Now it is possible to achieve local disturbance decoupling, by testing the conditions of the following theorem.

<u>Theorem</u>: For analytic control systems the state feedback disturbance decoupling problem is solvable locally (with β invertible) iff there exists a distribution Δ such that

Δ is (f,G) invariant

image $(P(x)) \subset \Delta_x \subset \text{Ker}(dh(x))$ //

This theorem together with the previous lemma provides a very useful tool for control synthesis to achieve disturbance decoupling.

The above theorem is due to Hirschorn [3] and independently Isidori, Krener, Gori-Georgi and Monaco [6]. Other variations due to Van der Schaft & Nijmeier are also known. See ([13]-[14])

5 Disturbance decoupling using momentum wheels:

In this section we show using the general theory outlined in section 4, that it is possible to design nonlinear feedback laws for 2 momentum wheels to decouple a part of the spacecraft attitude variable from internal disturbance torques.

5.1 Equations in local coordinates

We know from section 3 that the equations of motion are:

$$\begin{aligned} \dot{A} &= S(\omega)A \\ \dot{\omega} &= J^{-1}S(\omega)\, A\, h(0) + J^{-1}z \end{aligned} \tag{5.1.1}$$

Denoting the attitude matrix as

$$A = \begin{bmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{bmatrix}$$

we can rewrite (5.1.1) as

$$\begin{aligned}
\dot{r}_1 &= \omega_3 r_2 - \omega_2 r_3 \\
\dot{r}_2 &= -\omega_3 r_1 + \omega_1 r_3 \\
\dot{r}_3 &= \omega_2 r_1 - \omega_1 r_2 \\
\dot{s}_1 &= \omega_3 s_2 - \omega_2 s_3 \\
\dot{s}_2 &= -\omega_3 s_1 + \omega_1 s_3 \\
\dot{s}_3 &= \omega_2 s_1 - \omega_1 s_2 \\
\dot{t}_1 &= \omega_3 t_2 - \omega_2 t_3 \\
\dot{t}_2 &= -\omega_3 t_1 + \omega_1 t_3 \\
\dot{t}_3 &= \omega_2 t_1 - \omega_1 t_2 \\
\dot{\omega}_1 &= \omega_3\, \Omega_2^1 - \omega_2\, \Omega_3^1 + u_1 \\
\dot{\omega}_2 &= \omega_1\, \Omega_3^2 - \omega_3\, \Omega_1^2 + u_2 \\
\dot{\omega}_3 &= \omega_2\, \Omega_1^3 - \omega_1\, \Omega_2^3 + d
\end{aligned} \tag{5.1.2}$$

$$y = \begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix}$$

Where
$$\begin{aligned}
\Omega_2^1 &= \ell_1 r_2 + \ell_2 s_2 + \ell_3 t_2 \\
\Omega_3^1 &= \ell_1 r_3 + \ell_2 s_3 + \ell_3 t_3 \\
\Omega_3^2 &= \ell_4 r_3 + \ell_5 s_3 + \ell_6 t_3 \\
\Omega_1^2 &= \ell_4 r_1 + \ell_5 s_1 + \ell_6 t_1 \\
\Omega_1^3 &= \ell_7 r_1 + \ell_8 s_1 + \ell_9 t_1 \\
\Omega_2^3 &= \ell_7 r_2 + \ell_8 s_2 + \ell_9 t_2 \\
\Omega_1^1 &= \ell_1 r_1 + \ell_2 s_1 + \ell_3 t_1 \\
\Omega_2^2 &= \ell_4 r_2 + \ell_5 s_2 + \ell_6 t_2 \\
\Omega_3^3 &= \ell_7 r_3 + \ell_8 s_3 + \ell_9 t_3
\end{aligned} \tag{5.1.3}$$

Here (ℓ_1, ℓ_2, ℓ_3) is the normalized vector of total body angular momentum $h(0)$ (which remains invariant under internal disturbance torques).

5.2 Existence of an (f,G) invariant distribution

We have:

$$\underline{y} = \begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix} ; \ \underline{p} = e_{12} \ ; \ \ \underline{G} = [e_{10}, e_{11}]$$

where $\{e_1, \ldots, e_{12}\}$ denotes the standard basis in R^{12}.

We are looking for a distribution Δ such that:

$$\underline{p} \in \Delta \subset \text{Ker } dy = R^{12} / \text{Span } \{e_3, e_6, e_9\}$$

and Δ is locally (f,G) invariant:

$$[\underline{f}, \Delta] \subset \Delta + \text{Span } \{\underline{g_1 g_2}\}$$

$$[\underline{g_i}, \Delta] \subset \Delta + \text{Span } \{\underline{g_1 g_2}\}$$

H. Nijmeijer and van der Schaft [13] give an algorithm for computing the maximal (f,G) invariant distribution contained in a given distribution (Ker dy here); the same algorithm is given in a dual form by A. Isidori et al. in [6]. This algorithm requires us to solve systems of PDE's in a 12 x 12 + 12 = 156 dimensional space! We therefore have to take an alternative approach.

Consider the following distribution:

$$\Delta = \text{Span } \left\{ \underline{p} = e_{12} = X_1, \ X_2 = \begin{bmatrix} r_2 \\ -r_1 \\ 0 \\ s_2 \\ -s_1 \\ 0 \\ t_2 \\ -t_1 \\ 0 \\ \omega_2 \\ -\omega_1 \\ 0 \end{bmatrix} \right\}$$

Clearly $\underline{p} \in \Delta$ and $\Delta \subset$ Ker dy.

Now let us check that Δ is locally (f,G) invariant.

In all the following computations of Lie brackets, we need to compuate

$\dfrac{\partial \underline{f}}{\partial \underline{x}}$ which is given by:

$$\begin{bmatrix}
0 & \omega_3 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_3 & r_2 \\
-\omega_3 & 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & -r_1 \\
\omega_2 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_2 & r_1 & 0 \\
0 & 0 & 0 & 0 & \omega_3 & -\omega_2 & 0 & 0 & 0 & 0 & -s_3 & s_2 \\
0 & 0 & 0 & -\omega_3 & 0 & \omega_1 & 0 & 0 & 0 & s_3 & 0 & -s_1 \\
0 & 0 & 0 & \omega_2 & -\omega_1 & 0 & 0 & 0 & 0 & -s_2 & s_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 & -\omega_2 & 0 & -t_3 & t_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 & 0 & \omega_1 & t_3 & 0 & -t_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega_2 & -\omega_1 & 0 & -t_2 & t_1 & 0 \\
0 & \ell_1\omega_3 & -\ell_1\omega_2 & 0 & \ell_2\omega_3 & -\ell_2\omega_2 & 0 & \ell_3\omega_3 & -\ell_3\omega_2 & 0 & -\Omega_3^1 & \Omega_2^1 \\
-\ell_4\omega_3 & 0 & \ell_4\omega_1 & -\ell_5\omega_3 & 0 & \ell_5\omega_1 & -\ell_6\omega_3 & 0 & \ell_6\omega_1 & \Omega_3^2 & 0 & -\Omega_1^2 \\
\ell_7\omega_2 & -\ell_7\omega_1 & 0 & \ell_8\omega_2 & -\ell_8\omega_1 & 0 & \ell_9\omega_2 & -\ell_9\omega_1 & 0 & -\Omega_2^3 & \Omega_1^3 & 0
\end{bmatrix}$$

$\Delta = \text{Span}\ \{X_1,\ X_2\}$

$[\underline{f},\Delta] \subset \Delta + \text{Span}\ \{\underline{g}_1,\underline{g}_2\} \iff [\underline{f},\ X_1] \in \Delta + \text{Span}\ \{\underline{g}_1,\ \underline{g}_2\}$ (5.2.1)

$[\underline{f},\ X_2] \in \Delta + \text{Span}\ \{\underline{g}_1,\ \underline{g}_2\}$ (5.2.2)

$$[\underline{f},\ X_1] = \frac{\partial X_1}{\partial \underline{x}}\underline{f} - \frac{\partial f}{\partial \underline{x}} X_1 = \frac{-\partial f}{\partial \underline{x}} X_1 = \begin{bmatrix} -r_2 \\ +r_1 \\ 0 \\ -s_2 \\ s_1 \\ 0 \\ -t_2 \\ t_1 \\ 0 \\ -\Omega_2^1 \\ \Omega_1^2 \\ 0 \end{bmatrix}$$

Therefore $[\underline{f},\ X_1] = -X_2 + (\omega_2 - \Omega_2^1)\ \underline{g}_1 + (\Omega_1^2 - \omega_1)\ \underline{g}_2$

where $\underline{g}_1 = e_{10}$ and $\underline{g}_2 = e_{11}$

Hence $[\underline{f},\ \underline{X}_1] \in \Delta + \text{Span}\ \{\underline{g}_1\ \underline{g}_2\}$

Now, $$[\underline{f},\ X_2] = \frac{\partial X_2}{\partial \underline{x}}\underline{f} - \frac{\partial f}{\partial \underline{x}} \cdot X_2 \qquad (5.2.3)$$

$$\frac{\partial X_2}{\partial \underline{x}} = \begin{array}{c@{}c} & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array} & \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \end{array}$$

Hence:

$$[f,X_2] = (\omega_1\Omega_3^2 - \omega_3\Omega_1^2 + \omega_3\Omega_1^1 - \Omega_3^1\omega_1)\ \underline{g}_1 + (\omega_2\Omega_3^1 - \omega_3\Omega_2^1 - \omega_2\Omega_3^2 + \omega_3\Omega_2^2)\ \underline{g}_2 \in \Delta + \text{span}\ \{\underline{g}_1,\ \underline{g}_2\} \qquad (5.2.4)$$

Next we have to check that, for each x M, the statespace, the vectors, $[\underline{g}_1,\ X_1](x)$, $[\underline{g}_2,\ X_2](x)$, $[\underline{g}_1,\ X_2](x)$, $[\underline{g}_1,\ X_2](x)$ and $[\underline{g}_2,\ X_1](x)$ all belong to space $\Delta(x)$ + span $\{\underline{g}_1,\ g_2\}$. This can be verified by direct calculation that,

$$[\underline{g}_1,\ \underline{X}_1] = [g_2,\ X_1] = 0\ ;$$

and

$$[\underline{g}_1,\ X_2] = -e_{11}\ ;$$

$$[\underline{g}_2,\ X_2] = e_{10}\ .$$

This completes the verification that Δ is <u>locally (f, G) invariant.</u> It can be further <u>explicitly verified</u> [2], that.

(a). Δ is involutive and of rank 2.

(b). dim (span $\{\underline{g}_1,\ \underline{g}_2\}$) = 2 = constant on TSO(3) the tangent bundle of SO(3) = attitude x angular velocity space.

(c). dim ($\Delta \cap$ span $\{\underline{g}_1,\ \underline{g}_2\}$)

= 0 = also constant.

Thus the hypothese of the lemma of section 4 are satisfied. Hence there exists a feedback law $\alpha(x)$ and a nonlinear change of coordinates in the input space $\beta(x)$ such that Δ is (f,G) invariant i.e.,

$$[f + G\alpha,\ \Delta] \subset \Delta \qquad (5.2.5)$$

$$[(G\beta)_i,\ \Delta] \subset \Delta$$

$$i=1,2\ .$$

This implies, from the theorem of section 4, that the disturbance decoupling problem is solvable. In the next section we explicity compute $\alpha(\cdot)$ and $\beta(\cdot)$.

5.3 <u>Construction of decoupling feedback laws</u>

Summarizing what we already have:

$\Delta = \text{Span}\ \{X_1,\ X_2\}$

$$[\underline{f},\ X_1] = -X_2 + (\omega_2 - \Omega_2^1)\ \underline{g}_1 + (\Omega_1^2 - \omega_1)\ \underline{g}_2 \qquad \textbf{(5.3.1)}$$

$$[\underline{f},\ X_2] = (\omega_1\Omega_3^2 - \omega_3\Omega_1^2 + \omega_3\Omega_1^1 - \Omega_3\omega_1^1)\ \underline{g}_1 + (\omega_2\Omega_3^1 - \omega_3\Omega_2^1 + \omega_3\Omega_2^2 - \Omega_3^2\omega_2)\ \underline{g}_2 \qquad \textbf{(5.3.2)}$$

$$[g_1, X_1] = [g_2, X_1] = 0 \tag{5.3.3}$$

$$[g_1, X_2] = -g_2 \qquad [g_2, X_2] = g_1 \tag{5.3.4}$$

$$[X_1, X_2] = 0 \tag{5.3.5}$$

We want to find $\alpha(x) = \begin{bmatrix} \alpha_1(x) \\ \alpha_2(x) \end{bmatrix}$ and $\beta(x) =$

$$\begin{bmatrix} \lambda(x) & \mu(x) \\ \nu(x) & \sigma(x) \end{bmatrix}$$

such that Δ is invariant under the new dynamics:

$$\underline{\tilde{f}} = \underline{f} + \underline{G}\,\underline{\alpha}$$

and $\underline{\tilde{G}} = \underline{G}\,\underline{\beta}$.

$$\underline{G} = [\underline{g_1}, \underline{g_2}] = [e_{10}\ e_{11}]$$

$$\underline{G}\,\underline{\alpha} = [e_{10}\ e_{11}] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 e_{10} + \alpha_2 e_{11}$$

Therefore $\underline{f} + \underline{G}\,\underline{\alpha} = \underline{f} + \alpha_1 e_{10} + \alpha_2 e_{11}$ (5.3.6)

and we should have

$$[f + \alpha_1 e_{10} + \alpha_2 e_{11}, \Delta] \subset \Delta ,$$

or equivalently

$$[f + \alpha_1 e_{10} + \alpha_2 e_{11}, X_1] \in \Delta , \tag{5.3.7}$$

$$[f + \alpha_1 e_{10} + \alpha_2 e_{11}, X_2] \in \Delta . \tag{5.3.8}$$

(5.3.7) <=> $[f + \alpha_1 e_{10} + \alpha_2 e_{11}, X_1] =$

$$[f, X_1] - X_1(\alpha_1)\underline{g_1} - X_1(\alpha_2)\underline{g_2} \in \Delta \tag{5.3.9}$$

$$X_1(\alpha_1) = \frac{\partial \alpha_1}{\partial \omega_3}$$

$$X_1(\alpha_2) = \frac{\partial \alpha_2}{\partial \omega_3} . \tag{5.3.10}$$

Using equations (5.3.1), (5.3.9) and (5.3.10) we get:

$$\frac{\partial \alpha_1}{\partial \omega_3} = \omega_2 - \Omega_2^1 \tag{I}$$

$$-\frac{\partial \alpha_2}{\partial \omega_3} = \omega_1 - \Omega_1^2 \tag{II}$$

(5.3.8) <=> $[f + \alpha_1 \underline{g_1} + \alpha_2 \underline{g_2}, X_2] =$

$$[f, X_2] - (\alpha_1 + X_2(\alpha_2))\underline{g_2}$$

$$- (\alpha_2 - X_2(\alpha_1))\underline{g_2} \in \Delta \tag{11}$$

From (5.3.11) and (5.3.2) we get

$$\omega_3(\Omega_3^2 - \Omega_3^1) - \omega_3(\Omega_1^2 - \Omega_1^1) + \alpha_2 - X_2(\alpha_1) = 0 \tag{III}$$

$$\omega_3(\Omega_2^2 - \Omega_2^1) - \omega_2(\Omega_3^2 - \Omega_3^1) + \alpha_1 - X_2(\alpha_2) = 0 \tag{IV}$$

where $X_2(\alpha)$ is given by:

$$X_2(\alpha) = r_2 \frac{\partial \alpha}{\partial r_1} - r_1 \frac{\partial \alpha}{\partial r_2} + s_2 \frac{\partial \alpha}{\partial s_1} - s_1 \frac{\partial \alpha}{\partial s_2} +$$

$$t_2 \frac{\partial \alpha}{\partial s_2} - t_1 \frac{\partial \alpha}{\partial t_2} + \omega_2 \frac{\partial \alpha}{\partial \omega_1} - \omega_1 \frac{\partial \alpha}{\partial \omega_2} . \tag{5.3.12}$$

Therefore we seek to solve the system of PDE's (I)-(IV); which can be written as follows:

$$\frac{\partial \alpha_1}{\partial \omega_3} = \omega_2 - \Omega_2^1 \tag{I}$$

$$\frac{\partial \alpha_2}{\partial \omega_3} = -\omega_1 + \Omega_1^2 \tag{II}$$

$$X_2(\alpha_1) - \alpha_2 = \omega_1(\Omega_3^2 - \Omega_3^1) - \omega_3(\Omega_1^2 - \Omega_1^1) \tag{III}$$

$$X_2(\alpha_2) - \alpha_1 = \omega_3(\Omega_2^2 - \Omega_2^1) - \omega_2(\Omega_3^2 - \Omega_3^1) \tag{IV}$$

From (5.1.3) we get,

$$\Omega_3^2 - \Omega_3^1 = (\ell_4 - \ell_1)r_3 + (\ell_5 - \ell_2)s_3 + (\ell_6 - \ell_3)t_3 \tag{5.3.13}$$

$$\Omega_1^2 - \Omega_1^1 = (\ell_4 - \ell_1)r_1 + (\ell_5 - \ell_2)s_1 + (\ell_6 - \ell_3)t_1 \tag{5.3.14}$$

Similarly

$$\Omega_2^2 - \Omega_2^1 = (\ell_4 - \ell_1)r_2 + (\ell_5 - \ell_2)s_2 + (\ell_6 - \ell_3)t_2 \tag{5.3.15}$$

$$\Omega_3^2 - \Omega_3^1 = (\ell_4 - \ell_1)r_3 + (\ell_5 - \ell_2)s_3 + (\ell_6 - \ell_3)t_3 \tag{5.3.16}$$

From (5.3.13), (5.3.14) and (III) we get:

$$X_2(\alpha_1) - \alpha_2 = (\ell_4 - \ell_1)\,\dot{r}_2 + (\ell_5 - \ell_2)\,\dot{s}_2 + (\ell_6 - \ell_3)\,\dot{t}_2 \tag{5.3.17}$$

From (5.3.15), (5.3.16) and (IV) we get:

$$X_2(\alpha_2) - \alpha_1 = (\ell_4 - \ell_1)\dot{r}_1 + (\ell_5 - \ell_2)\dot{s}_1 + (\ell_6 - \ell_3)\dot{t}_1 \quad (5.3.18)$$

Now, let us make the following change of variables:

$$\alpha_1 = (\omega_2 - \ell_1 r_2 - \ell_2 s_2 - \ell_3 t_2)\omega_3 + A_1(r_1, r_2, \ldots \omega_2) \quad (5.3.19)$$

$$\alpha_2 = (-\omega_1 + \ell_4 r_1 + \ell_5 s_1 + \ell_6 t_1)\omega_3 + A_2(r_1, r_2, \ldots \omega_2) \quad (5.3.20)$$

Define the new variables:

$$\sigma_1 = \ell_4 - \ell_1 \qquad \sigma_2 = \ell_5 - \ell_2 \qquad \sigma_3 = \ell_6 - \ell_3 \quad (5.3.21)$$

After further manipulations, we get the following system:

$$X_2(A_1) - A_2 = \sigma_1\omega_1 r_3 + \sigma_2\omega_1 s_3 + \sigma_3\omega_1 t_3 \quad (V)$$

$$X_2(A_2) + A_1 = -\sigma_1\omega_2 r_3 - \sigma_2\omega_2 s_3 + \sigma_3\omega_2 t_3 \quad (VI)$$

One can check that:

$$A_1 = -\sigma_1\omega_2 r_3 - \sigma_2\omega_2 s_3 - \sigma_3\omega_2 t_3 \quad (22)$$

$$A_2 = 0 \quad (23)$$

is a solution to the system (V), (VI).

This leads to the following expressions for α_1 and α_2:

$$\alpha_1 = \omega_2\omega_3 - \ell_1 r_2\omega_3 - \ell_2 s_2\omega_3 - \ell_3 t_2\omega_3 - \sigma_1\omega_2 r_3 - \sigma_2\omega_2 s_3 - \sigma_3\omega_2 t_3 \quad (5.3.24)$$

$$\alpha_2 = -\omega_1\omega_3 + \ell_4 r_1\omega_3 + \ell_5 s_1\omega_3 + \ell_6 t_1\omega_3 \quad (5.3.25)$$

Furthermore α_1 and α_2 satisfy equations (I) and (II):

From (5.3.24) we have

$$\frac{\partial\alpha_1}{\partial\omega_3} = \omega_2 - (\ell_1 r_2 + \ell_2 s_2 + \ell_3 t_2) = \omega_2 - \Omega_2^1$$

From (5.3.25) we have

$$\frac{\partial\alpha_2}{\partial\omega_3} = -\omega_1 + (\ell_4 r_1 + \ell_5 s_1 + \ell_6 t_1) = -\omega_1 + \Omega_1^2.$$

Now, let us compute β which satisfies:

$$[(G\beta)_i, \Delta] \subset \Delta \quad , i=1,2, \quad (5.3.26)$$

$$G\beta = [e_{10}\ e_{11}]\begin{bmatrix}\lambda & \mu \\ \nu & \sigma\end{bmatrix} = [\lambda e_{10} + \nu e_{11}, \mu e_{10} + \sigma e_{11}] \quad (5.3.27)$$

Therefore $\quad [\lambda e_{10} + \nu e_{11}, \Delta] \subseteq \Delta \quad (5.3.28)$

$$[\mu e_{10} + \sigma e_{11}, \Delta] \subset \Delta \quad (5.3.29)$$

the pairs (λ,ν) and (μ,σ) play a symmetric role and therefore will satisfy the same system of partial differential equations.

$$(5.3.28) \Rightarrow [\lambda e_{10} + \nu e_{11}, X_1] \in \Delta$$

but $[\lambda e_{10} + \nu e_{11}, X_1] = -X_1(\lambda)e_{10} - X_1(\nu)e_{11}$

Hence $\quad X_1(\lambda) = 0 \quad (5.3.30)$

$$X_1(\nu) = 0 \quad (5.3.31)$$

$$(5.3.28) \Rightarrow [\lambda e_{10} + \nu e_{11}, X_2] = (\nu - X_2(\lambda))e_{10} - (\lambda + X_2(\nu))e_{11} \in \Delta$$

Therefore $\quad X_2(\lambda) - \nu = 0 \quad (5.3.32)$

$$X_2(\nu) - \lambda = 0 \quad (5.3.33)$$

(μ,σ) satisfy the same system of equations (5.3.30)-(5.3.33). We note that (r_1, r_2), (s_1, s_2) and (t_1, t_2) are all solutions to the system of PDE's (5.3.30)-(5.3.33).

Now, we summarize our results in the following theorem.

Theorem : Consider the momentum wheels attitude control system of equation 5.1.1. The feedback law:

$$u_1 = (\omega_2\omega_3 - \ell_1 r_2\omega_3 - \ell_2 s_2\omega_3 - \ell_3 t_2\omega_3 - \sigma_1\omega_2 r_3 - \sigma_2\omega_2 s_3 - \sigma_3\omega_2 t_3) + r_1 v_1 + s_1 v_2$$

$$u_2 = (-\omega_1\omega_3 + \ell_4 r_1\omega_3 + \ell_5 s_1\omega_3 + \ell_6 t_1\omega_3) + r_2 v_1 + s_2 v_2$$

decouples the last row (r_3, s_3, t_3) of the attitude matrix from a disturbance acting on an internal disturbance torque acting along the ω_3 axis.

6. Final Remarks:

We have shown how differential geometric methods contribute to the solution of two types of problems arising in multibody spacecraft - spin stabilization and disturbance decoupling. The tools used here are expected to play an essential role in rigorous analytical investigations of more complex multibody spacecraft and related control synthesis questions.

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