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Optimal Control and Poisson Reduction

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Optimal Control and Poisson Reduction

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Abstract:

In this paper we make explicit a reduction of G -invariant optimal control problems on a Lie group G .

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1. Introduction

Let G be a finite dimensional Lie group. In this paper, we consider left-invariant control systems on G . To fix notation, let $L_g : G \rightarrow G$, $h \mapsto g \cdot h$, denote the left translation by $g \in G$, and let $T_h L_g : T_h G \mapsto T_{gh} G$ denote its tangent (i.e linearization). Let \mathcal{G} denote the Lie algebra of G (i.e $T_e G$ the tangent space at identity $e \in G$). A vector field $g \mapsto v_g \in TG$ is left-invariant if $v_g = T_e L_g \cdot \xi$ for some fixed $\xi \in \mathcal{G}$. A left-invariant control system is defined by letting ξ be a (controlled) curve in \mathcal{G} . It takes the form

$$\dot{g} = T_e L_g \cdot \xi_u \quad (1)$$

where each control $u(\cdot)$ determines a curve $\xi_{u(\cdot)} \subset \mathcal{G}$. Here we limit ourselves to vector valued control functions $u(\cdot)$.

$$\xi_u = \xi_0 + \sum_{i=1}^m u_i(t) \xi_i \quad (2)$$

where $u_i(t) \in \mathbb{R}$, $\{\xi_0, \xi_1, \dots, \xi_m\}$ spans an $(m + 1)$ -dimensional subspace \mathcal{H} of \mathcal{G} , $m + 1 \leq n = \dim(\mathcal{G})$.

Consider an optimal control problem of the form

$$\underset{u(\cdot)}{\text{Min}} \int_0^T L(u) dt \quad (3)$$

subject to the condition that $u(\cdot)$ steers (1) - (2) from g_0 at $t = 0$ to g_1 at $t = T$. (T may be fixed or free. For simplicity we restrict T to be fixed in this paper.)

Clearly, the Lagrangian L is G -invariant (-does not depend on g).

It is the content of the *maximum principle* that optimal curves in G are base integral curves of a hamiltonian vector field on T^*G . To be more precise, let $\tau_G : TG \rightarrow G$ and $\tau_G^* : T^*G \rightarrow G$ be bundle projections.

Define

$$\begin{aligned} H^\lambda &= H^\lambda(\alpha_g, u) \\ &= -\lambda L(u) + \langle \alpha_g, T_e L_g \cdot \xi_u \rangle \end{aligned} \quad (4)$$

where $\lambda = 1$ or 0 , and $\alpha_g \in T^*G$.

Maximum Principle

Let u_{opt} be a minimizer of (3) and let g be the corresponding trajectory of (1). Then, $g(t) = \tau_G^*(\alpha_g(t))$ for an integral curve α_g of the hamiltonian vector field $X_{H^\lambda}^{u_{opt}}$ defined for $t \in [0, T]$ such that:

(a) If $\lambda = 0$ then α_g is not the zero section of T^*G on $[0, T]$.

(b) $H^\lambda(\alpha_g, u_{opt}) = \sup_{u \in U} H^\lambda(\alpha_g, u)$ for t a.e. in $[0, T]$. Here U is the space of values of controls $= \mathbb{R}^m$ in this paper.

(c) If the terminal time T is fixed then $H^\lambda(\alpha_g, u_{opt}) = \text{constant}$, and if T is free then $H^\lambda(\alpha_g, u_{opt}) = 0 \quad \forall t \in [0, T]$. Trajectories corresponding to $\lambda = 0$ are called abnormal extremals and they occur often but are ruled out under suitable hypotheses. *We are concerned solely with regular extremals ($\lambda = 1$) in this paper.* In that case, since $U = \mathbb{R}^m$, optimal controls for (3) subject to (1) and (2) satisfy

$$\begin{aligned} \frac{-\partial L}{\partial u_i} + \frac{\partial}{\partial u_i} \langle \alpha_g, T_e L_g \cdot \xi_u \rangle &= 0. \\ i &= 1, 2, \dots, m. \end{aligned} \quad (5)$$

From (2),

$$\begin{aligned} \langle \alpha_g, T_e L_g \cdot \xi_u \rangle &= \langle \alpha_g, T_e L_g \cdot (\xi_0 + \sum_{i=1}^m u_i \xi_i) \rangle \\ &= \langle T_e L_g^* \cdot \alpha_g, \xi_0 + \sum_{i=1}^m u_i \xi_i \rangle \\ &= \langle \mu, \xi_0 \rangle + u_i \sum_{i=1}^m \langle \mu, \xi_i \rangle \end{aligned} \quad (6)$$

where $\mu = TeL_g^* \cdot \alpha_g \in \mathcal{G}^*$. From (5) and (6)

$$\begin{aligned} \frac{-\partial L}{\partial u_i} + \langle \mu, \xi_i \rangle &= 0, \\ i &= 1, 2, \dots, m. \end{aligned} \tag{7}$$

Observe that, from (6) and (7) the pre-hamiltonian $H^{\lambda=1}$ and the hamiltonian $H^{\lambda=1}$ are G -invariant. More explicitly, suppose $L(u) = \frac{1}{2} \sum_{i=1}^m I_i u_i^2$, the constants $I_i > 0$. Then, the optimal u_i are given by

$$u_i = \frac{\langle \mu, \xi_i \rangle}{I_i} \tag{8}$$

and the hamiltonian on T^*G is

$$H = \langle \mu, \xi_0 \rangle + \frac{1}{2} \sum_{i=1}^m \frac{\langle \mu, \xi_i \rangle^2}{I_i} \tag{9}$$

clearly the hamiltonian H is G -invariant. This brings us to the main (and *essentially only point*) of this paper. The hamiltonian vector field X_H on T^*G corresponding to the hamiltonian H can be (left-) reduced to a hamiltonian vector field X_h on \mathcal{G}^* . The latter is hamiltonian in a non-canonical (Lie-Poisson) sense. Integral curves of X_H can be reconstructed from integral curves of X_h by quadrature. Thus questions about explicit solvability (or integrability) of X_H are turned into corresponding questions about X_h .

The process by which one passes from X_H to X_h , known as Poisson reduction, is well-known in geometric mechanics. Since \mathcal{G}^* is a vector space, we thus conveniently bypass computations with the symplectic structure on T^*G . In his papers Jurdjevic works with the symplectic structure on T^*G although one does see reduction “done by hand” in special examples.

The paper of Faybusovich is closest to the point-of-view of the present paper and is in some ways broader in scope. Limiting ourselves to G -invariant optimal control problems as here, we hope to make things a bit more explicit than is perhaps in [2].

In the next few sections we collect together some basic facts regarding Poisson manifolds, symmetry and reduction and carry out the reduction step for the optimal control problem (1)-(3).

2. Poisson Manifolds, Symmetry Groups and Reduction

A smooth manifold P is said to be a Poisson manifold if it comes equipped with an \mathbb{R} -bilinear map (Poisson structure),

$$\{\cdot, \cdot\}_P : C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P)$$

satisfying the axioms:

$$\begin{aligned} (i) \quad & \{\phi, \psi\}_P = -\{\psi, \phi\} \\ (ii) \quad & \{\phi, \{\psi, \chi\}_P\}_P + \{\psi, \{\chi, \phi\}_P\}_P + \{\chi, \{\phi, \psi\}_P\}_P = 0 \\ (iii) \quad & \{\phi\psi, \chi\}_P = \psi \{\phi, \chi\}_P + \phi \{\psi, \chi\}_P \end{aligned}$$

Associated to a Poisson structure is a unique twice-contravariant, skew-symmetric, tensor field Λ on P on such that

$$\{\phi, \psi\}_P = \Lambda(d\phi, d\psi). \tag{10}$$

For a proof see p. 109 of [8]. The tensor Λ defines a vector-bundle morphism,

$$\begin{aligned} \Lambda^\# : T^*P &\rightarrow TP \\ \alpha_x &\mapsto \Lambda^\#(\alpha_x) \in T_xP \end{aligned}$$

satisfying,

$$\beta_x (\Lambda^\# (\alpha_x)) = \Lambda(x) (\alpha_x, \beta_x) \text{ for all } \alpha_x, \beta_x \in T_xP.$$

The rank of the Poisson structure at $x \in P$ is defined to be the rank of the Poisson tensor Λ at x . This is simply the rank of the (characteristic) distribution $C = \Lambda^\#(T^*P) \subset TP$ at the point x . The rank may vary on P . It is a theorem of Kirillov [6] that $\Lambda^\#(T^*P)$ defines a generalized foliation on P such that through each point $x \in P$, passes a leaf

carrying a unique symplectic structure that makes the injection map of the leaf a Poisson morphism. (See Weinstein [13] and Liebermann-Marle [8]). Thus a Poisson manifold is a *union of symplectic leaves*.

A function $\phi \in C^\infty(P)$ is called a *Casimir function* if $\{\phi, \psi\}_P = 0 \forall \psi \in C^\infty(P)$. Casimir functions are constant on symplectic leaves.

Let G be a Lie group and let $\Psi : G \times P \rightarrow P, (g, x) \mapsto \Psi_g(x)$ be a group action such that, $\Psi_g(\cdot)$ is a Poisson morphism for every $g \in G$. Further suppose that the action is proper and free. Then the quotient manifold P/G is smooth and carries the Poisson structure $\{\cdot, \cdot\}_{P/G}$ induced from the one on P satisfying,

$$\{\phi, \psi\}_{P/G \circ \pi} = \{\phi \circ \pi, \psi \circ \pi\}_P,$$

where $\pi : P \rightarrow P/G$ is the canonical projection. By construction, it is a Poisson morphism.

Now G -invariant dynamics on P induces dynamics on P/G . To see this, let $H : P \rightarrow \mathbb{R}$ be a G -invariant hamiltonian function on P , i.e.,

$$H(\Psi_g(x)) = H(x) \quad \forall g \in G.$$

Define the (hamiltonian) vector field X_H on P by requiring,

$$X_H \phi = \{\phi, H\} \quad \forall \phi \in C^\infty(P).$$

The hamiltonian H descends to $h : P/G \rightarrow \mathbb{R}$ and determines the *reduced* hamiltonian dynamics X_h on P/G by

$$X_h \bar{\phi} = \{\bar{\phi}, h\}_{P/G} \quad \forall \bar{\phi} \in C^\infty(P/G).$$

Here $h([x]) = H(x)$ for any equivalence class $[x]$ in P/G .

From the definition of the characteristic distribution on a Poisson manifold, it follows that the hamiltonian vector field X_h leaves invariant the symplectic leaves of P/G . Thus

any Casimir function on P/G is an integral of motion for X_h . The integral curves of X_H project under π to integral curves of X_h .

The process sketched out so far is the essence of Poisson reduction. See [9, 10] for details and generalizations.

For our purposes the main examples of Poisson manifolds are T^*G and \mathcal{G}^* . The Poisson structure on T^*G is given by the canonical symplectic structure ω on T^*G as,

$$\{\phi, \psi\}_{T^*G} = \omega(X_\phi, X_\psi) \quad (11)$$

where ω is the canonical symplectic form on T^*G (c.f. Abraham-Marsden p. 315).

The second example of interest to us is the space \mathcal{G}^* the linear dual of a Lie algebra \mathcal{G} . The space \mathcal{G}^* carries the Lie-Berezin- Kirillov - Kostant - Souriau Poisson structure (s), defined by

$$\{\phi, \psi\}_\mp(\mu) = \mp \left\langle \mu, \left[\frac{\delta\phi}{\delta\psi}, \frac{\delta\psi}{\delta\mu} \right] \right\rangle, \quad (12)$$

where $\phi, \psi \in C^\infty(\mathcal{G}^*)$ and $\mu \in \mathcal{G}^*$. Here the variational derivative $\frac{\delta}{\delta\mu}$ is defined via the Frechet derivative:

$$D\phi(\mu) \cdot \delta = \left\langle \frac{\delta\phi}{\delta\mu}, \delta\phi \right\rangle$$

for $\delta\phi \in \mathcal{G}^*$. By finite dimensionality, $\mathcal{G}^{**} = \mathcal{G}$ and $\frac{\delta\phi}{\delta\mu} \in \mathcal{G}$.

The minus (plus) bracket is obtained by viewing \mathcal{G}^* as the left (right) Poisson reduction of T^*G by G , as explained further in the next section.

3. Lie - Poisson Reduction and the Maximum Principle

Recall that the tangent and cotangent bundles of Lie groups are trivial as vector bundles, i.e.

$$TG \simeq G \times \mathcal{G} \text{ and } T^*G \cong G \times \mathcal{G}^*.$$

The trivializations are made explicit by tangents of left and right translation operators. Thus

$$\begin{aligned} (id, \lambda) : T^*G &\rightarrow G \times \mathcal{G}^* \\ \alpha_g &\mapsto (g, T_e L_g^* \alpha_g) \\ (id, \rho) : T^*G &\rightarrow G \times \mathcal{G}^* \\ \alpha_g &\mapsto (g, T_e R_g^* \alpha_g) \end{aligned}$$

are the corresponding Poisson diffeomorphisms. In the case of $G = SO(3)$ the rotation group, these diffeomorphisms correspond to passage to body and space coordinates respectively, in rigid body mechanics.

Observe that the canonical symplectic structure on T^*G is G -invariant and is best expressed in one of the trivializations (A & M p. 315). In each case the quotient $T^*G/G \simeq \mathcal{G}^*$.

Proposition (Lie-Poisson Reduction):

The quotient maps

$$\begin{aligned} \lambda : T^*G &\rightarrow \mathcal{G}^* \\ \rho : T^*G &\rightarrow \mathcal{G}^* \end{aligned}$$

are Poisson homomorphisms if for λ we take the minus Lie-Poisson structure on \mathcal{G}^* and for ρ we take the plus Lie Poisson structure on \mathcal{G}^* .

Proof. It is a calculation using the expression of ω on T^*G in one of the two trivializations, and using the definition of quotient Poisson structure as in section 2. See for instance [9] [10]. ■

Denote the space \mathcal{G}^* with the minus (plus) Poisson structure (12) as \mathcal{G}_-^* (\mathcal{G}_+^*). From the discussion in section 2, every left - invariant hamiltonian on T^*G gives rise to a reduced dynamics on \mathcal{G}_-^* . (For our optimal control problem, left reduction is of interest.)

Suppose $h : \mathcal{G}_-^* \rightarrow \mathbb{R}$ is a reduced hamiltonian function. Let $\{X_1, \dots, X_n\}$ be a basis for the Lie algebra \mathcal{G} . Let $\{X_1^b, \dots, X_n^b\}$ be the dual basis for \mathcal{G}^* . Thus $\langle X_i^b, X_j \rangle = \delta_j^i$ the Kronecker Symbol. Any $\mu \in \mathcal{G}_-^*$ can be expressed as $\mu = \sum_{i=1}^n \mu_i X_i^b$. The differential equations satisfied by the coordinate functions are

$$\dot{\mu}_i = \{\mu_i, h\}_- \quad i = 1, 2, \dots, n \quad (13)$$

where the Poisson bracket has to be computed from (12). Explicitly

$$\dot{\mu}_i = - \sum_{j=1}^n \sum_{k=1}^n \mu_k \Gamma_{ij}^k \frac{\delta h}{\delta \mu_j}, \quad i = 1, 2, \dots, n. \quad (14)$$

Here Γ_{ij}^k are the structure constants of \mathcal{G} in the basis $\{X_1, \dots, X_n\}$.

By the properties of the Poisson bracket it is immediate that h is constant along trajectories of (13). Additionally, the Casimir functions on \mathcal{G}_- are also constant along trajectories of (13). For $\mathcal{G} = SO(3)$ the Lie algebra of skew symmetric matrices, any Casimir function is of the form $\Phi(\mu_1^2 + \mu_2^2 + \mu_3^2)$. In general there may not be any nontrivial (i.e. non-constant) Casimir functions. (The Lie algebra of the affine group or $ax + b$ group is an example.)

Returning to the optimal control problem of this paper, since the hamiltonian H in (9) is already expressed as function on \mathcal{G}^* , we note that the reduced hamiltonian is

$$h = \langle \mu, \xi_0 \rangle + \frac{1}{2} \sum_{i=1}^m \frac{\langle \mu, \xi_i \rangle^2}{I_i}. \quad (15)$$

We have in effect shown the following reduction of the maximum principle.

Theorem. Consider the optimal control problem,

$$\underset{u}{Min} \int_0^T \frac{1}{2} \sum_{i=1}^m I_i u_i^2 dt$$

subject to

$$\dot{g} = T_e L_g \cdot \left(\xi_0 + \sum_{i=1}^m u_i(t) \xi_i \right),$$

$$g(0) = g_0 \text{ and } g(T) = g_1.$$

Then every regular extremal is given by

$$u_i = \langle \mu, \xi_i \rangle / I_i,$$

where μ is an integral curve of the vector field X_h on \mathcal{G}_-^* corresponding to the hamiltonian

$$h = \langle \mu, \xi_0 \rangle + \frac{1}{2} \sum_{i=1}^m \frac{\langle \mu, \xi_i \rangle^2}{I_i}$$

and the Poisson bracket $\{\cdot, \cdot\}_-$ on \mathcal{G}_-^* is given by (12). In coordinates on \mathcal{G}_-^* the integral curves satisfy the ordinary differential equations (13)-(14). \blacksquare

4. Example: The Unicycle.

Consider the unicycle equations

$$\dot{x} = \cos(\phi) u_2$$

$$\dot{y} = \sin(\phi) u_2$$

$$\dot{\phi} = u_1$$

These equations are equivalent to the nonholonomic constraint $\dot{x} \sin \phi - \dot{y} \cos \phi = 0$.

Set,

$$g = \begin{bmatrix} \cos \phi & -\sin \phi & x \\ \sin \phi & \cos \phi & y \\ 0 & 0 & 1 \end{bmatrix}.$$

clearly $g \in SE(2)$ the rigid motion group of the plane. Further the unicycle equations take the form

$$\dot{g} = g \cdot (X_1 u_1 + X_2 u_2) \tag{16}$$

where,

$$\begin{aligned}
X_1 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
X_2 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
\tag{17}$$

clearly this is a left-invariant system in $SE(2)$. Further, since the Lie algebra of $SE(2) = se(2) = span \{X_1, X_2, X_3\}$, where

$$X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $[X_1, X_2] = X_3$, (also $[X_1, X_3] = -X_2$ and $[X_2, X_3] = 0$), it follows that we have a controllable system on $SE(2)$. (This way of looking at the unicycle was pointed out by Naomi Leonard.)

Next consider the optimal control problem

$$min \int_0^T (u_1^2 + u_2^2) dt$$

subject to end-point conditions. By our theorem, regular extremals are given by integral curves of a reduced hamiltonian on $se(2)_-^*$. The hamiltonian is in fact, in the coordinates corresponding to the dual basis $\{X_1^b, X_2^b, X_3^b\}$, given by $h = (\mu_1^2 + \mu_2^2)/2$. The Poisson bracket of two functions ϕ and ψ is given by

$$\{\phi, \psi\}_- = \nabla \phi^T \Lambda(\mu) \nabla \psi$$

where,

$$[\Lambda(\mu)]_{ij} = - \sum_{k=1}^3 \Gamma_{ij}^k \mu_k.$$

Explicitly,

$$\Lambda = \begin{pmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & 0 \\ -\mu_2 & 0 & 0 \end{pmatrix}. \quad (18)$$

The reduced hamilton's equations are (from (14)),

$$\begin{aligned} \dot{\mu}_1 &= -\mu_2\mu_3 \\ \dot{\mu}_2 &= \mu_1\mu_3 \\ \dot{\mu}_3 &= \mu_1\mu_2. \end{aligned} \quad (19)$$

The Casimir functions are of the form $\Phi = \Phi(\mu_2^2 + \mu_3^2)$, (equivalently $\nabla\Phi$ is in the kernel of Λ). The level sets of Casimir functions (i.e symplectic leaves in \mathcal{G}_-^*) are right circular cylinders $\{\mu : \mu_2^2 + \mu_3^2 = c\}$. Integral curves of (19) are intersections of level sets of h , also cylinders $\{\mu : \mu_1^2 + \mu_2^2 = 2h\}$, with the symplectic leaves. Thus the phase curves on symplectic leaves are as in Figure 1.

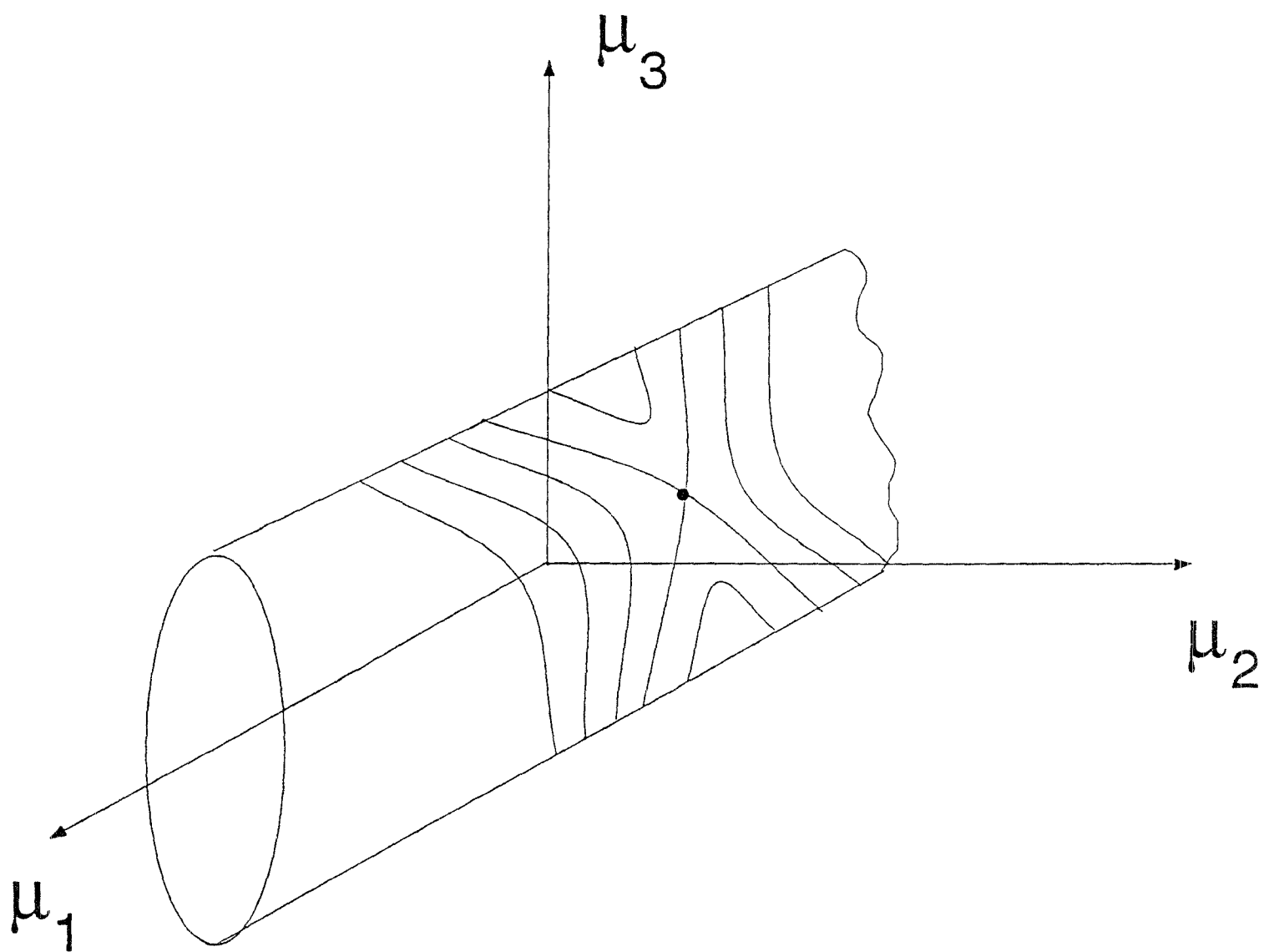


Figure 1

Equations (19) are explicitly solvable by elliptic functions. To see this, note that,

$$\ddot{\mu}_2 + (2H + C) \mu_2 - 2/\mu_2^3 = 0. \quad (20)$$

This is the equation of an anharmonic oscillator with quartic potential term. The general solution to (20) is given by,

$$\mu_2(t) = \beta \operatorname{Sn}(\lambda(t - t_0), k)$$

where $\operatorname{Sn}(u, k)$ is Jacobi's elliptic sine function, λ s.t. $\lambda^2 < (2H + C) < 2\lambda^2$, t_0 is arbitrary, $k^2 = \frac{2H + C}{\lambda^2} - 1$, $\beta^2 = 2H + C - \lambda^2$. Then μ_1 and μ_3 are determined from $\mu_1 = \sqrt{2H - \mu_2^2}$; $\mu_3 = \sqrt{C - \mu_2^2}$.

The optimal controls are given by

$$u_1 = \mu_1; u_2 = \mu_2.$$

It can be shown that in this example, there are no abnormal extremals since the *strong bracket-generating condition* holds (c.f. Strichartz [12])

5. Conclusion

We have worked out explicitly the Poisson reduction of certain G -invariant optimal control problems on Lie groups. The approach presented here yields an algorithm for constructing regular extremals.

6. Notes

This paper is meant to be used as a set of notes for a mini-lecture at the Workshop on Mechanics, Holonomy and Control, IEEE CDC, Dec 14 1993. The main ideas of this paper are already to be found in the papers of Faybusovich referred to below. He used the language of collective hamiltonians and we use the language of Poisson reduction. We are also a bit more fussy about right and left reduction and the different reduced Poisson structures one associates with them. In the papers of Jurdjevic, the reduction is carried out in various examples by hand, but Jurdjevic tends to work with the symplectic structures.

These papers of Jurdjevic and the references therein contain many additional examples of explicitly solvable optimal control problems. Our own enthusiasm for Lie-Poisson structures dates back at least to our work [7] on the dual spin problem. This and later collaborations have influenced our thinking. A special thanks is extended to Leonid Faybusovich for useful and enthusiastic discussions.

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