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Geometric Phases, and Optical Reconfiguration for Multibody Systems

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Abstract

Relative Motion in a system of coupled rigid bodies can yield global reorientation (or phase shift). We give a formula to compute such a phase shift and interpret the same in geometric terms. The theory of connections in principal bundles provides the proper setting for questions of the type addressed in this paper. A related optimal control problem leads to singular riemannian geometry.

1. Background

In 1987, Jair Koiller introduced me to the work of M.V. Berry [1] and B. Simon [15] on geometric phases in classical and quantum physics. Inspired by his remarks, I obtained formulas for analogous geometric phases in multibody systems. It became apparent, from conversations with Jerrold Marsden, that principal connections were involved. Around this time, A. Shapere was completing his thesis on gauge kinematics of deformable bodies under F. Wilczek [12]. In the hands of Marsden, Montgomery and Ratiu, the gauge-theoretic links between phases and the reconstruction problem in hamiltonian mechanics [10] [6] [8] have become very clear. In the present paper, I give a "bare hands" derivation of the phase shift formula for coupled planar rigid bodies, formulate an optimal control problem, and present the geometric picture that generalizes the example of this paper.

My understanding of the gauge-theoretic framework for reconstruction phases, owes a great deal to the many conversations I have had on this subject with Jerrold Marsden, Richard Montgomery and Tudor Ratiu. Thanks are also due to Al Shapere and Frank Wilczek for stimulating discussions and references. This work is an off-shoot of the program to understand the dynamics of multibody systems begun in collaboration with Marsden, Sreenath and others.

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2. Planar N-body Dynamics

Consider a system of planar rigid laminae connected by single degree of freedom pin joints. The bodies are free to float and move freely in space (in the absence of external forces and ignoring self-collisions). See Figure 1.

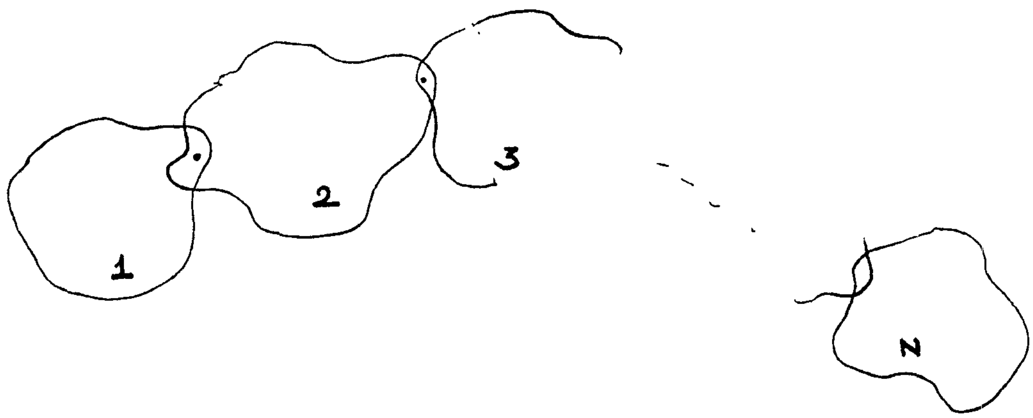

Figure 1

The instantaneous configuration of such a system relative to the center-of-mass of the assembly is given by the N-tuple of absolute orientations of frames rigidly affixed to each body, say at its center-of-mass. We denote it as $q = (q_1, \cdots, q_N) \in \mathbf{T}^N$, the N-torus. Let $\omega_i \underline{\triangle} \dot{q}_i$ denote the absolute angular velocities and $\omega = (\omega_1, \cdots, \omega_N)^T$. The joints are assumed to be articulated via motors with drive torques u_i, $i = 1, 2, \cdots, (N - 1)$. It can be shown [16] [18] that the N-body system is governed by a Hamiltonian control system:

$$\begin{aligned} \dot{q} &= \mathbf{J}(q)^{-1}\, p \\ \dot{p} &= -\frac{\partial}{\partial q}\frac{1}{2}\,\langle p, \mathbf{J}(q)^{-1}\, p\rangle \; + \; Bu, \end{aligned} \tag{2.1}$$

where $\mathbf{J}$ is a configuration-dependent $N \times N$ matrix that defines the free Hamiltonian,

$$H = \frac{1}{2}\,\langle p, \mathbf{J}^{-1}(q)\, p\rangle, \tag{2.2}$$

and the $N \times (N-1)$ matrix B satisfies,

$$B_{ij} = \begin{cases} 1 & i = j \\ -1 & i = j + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The matrix $\mathbf{J}$ depends only on the differences of absolute angles and is hence invariant under the S^1 action, $(q_1, \cdots, q_N) \rightarrow (q_1 + \phi, q_2 + \phi, \cdots, q_N + \phi)$, where $\phi \in S^1$. The angular momentum $\mu = \sum_{i=1}^{N} p_i$ is conserved (control torques are *internal*).

The space of equivalence classes of configurations under the *diagonal* S^1 action on $\mathbf{T}^N$ is referred to as the (labelled) *shape space*. It is the space $\mathbf{T}^{N-1}$ of joint angles $\theta_i = q_{i+1} - q_i$, $i = 1, 2, \cdots, N-1$. Poisson reduction by S^1 yields the controlled dynamics on $P = \mathbf{T}^{N-1} \times \mathbb{R}^N$,

$$
\begin{aligned}
\dot{\theta}_i &= \frac{\partial H}{\partial p_{i+1}} - \frac{\partial H}{\partial p_i} \qquad i = 1, 2, \cdots, N-1. \\
\dot{p}_1 &= \frac{\partial H}{\partial \theta_1} + u_1 \\
\dot{p}_2 &= -\frac{\partial H}{\partial \theta_1} + \frac{\partial H}{\partial \theta_2} - u_1 + u_2 \\
&\cdot \\
&\cdot \\
p_{N-1} &= -\frac{\partial H}{\partial \theta_{N-2}} + \frac{\partial H}{\partial \theta_{N-1}} - u_{N-2} + u_{N-1} \\
\dot{p}_N &= -\frac{\partial H}{\partial \theta_N} - u_{N-1}.
\end{aligned}
\tag{2.4}
$$

For any choice of controls, the motions are confined to the *symplectic leaves* $\mu = \sum_{i=1}^{N} p_i =$ constant, in P. Much of this is by now standard and may be found in [16] [18] [11]. For Poisson reduction, see [7] [5]. We pose,

The Reconstruction Problem: Given a motion in shape space (symplectic leaf) reconstruct the motion in configuration space (unreduced phase space).

A partial solution to the reconstruction problem can be given by considering the *splitting*:

$$\omega \triangleq \dot{q} = \omega_1 e + M \dot{\theta} \tag{2.5}$$

where $M = [M_{ij}]$ is the $N \times (N-1)$ matrix defined by,

$$M_{ij} = \begin{cases} 0, & i = 1 \\ 1, & i > j \\ 0, & \text{otherwise,} \end{cases}$$

and $e = (1, 1, \cdots, 1)^T$ is a column vector in $\mathbb{R}^N$. Then, using $\mu = e \cdot \mathbf{J}\omega$, we get,

$$\omega_1 = \frac{\mu}{e \cdot \mathbf{J} e} - \frac{e \cdot \mathbf{J} M \dot{\theta}}{e \cdot \mathbf{J} e}. \tag{2.6}$$

Integrating both sides of (2.6) over a path γ in the shape space, we get a formula for the absolute reorientation or phase shift of the body numbered 1:

$$\boxed{\begin{aligned}\triangle q_1 &= \int_\gamma \frac{\mu}{e \cdot \mathbf{J}\, e}\, dt - \int_\gamma \frac{e \cdot \mathbf{J}\, M\, d\theta}{e \cdot \mathbf{J} e}, \\ &= \frac{2}{\mu} \int_\gamma V_\mu\, (\theta(t))\; dt - \int_\gamma \frac{e \cdot \mathbf{J}\, M\, d\theta}{e \cdot \mathbf{J}\, e}.\end{aligned}} \tag{2.7}$$

The first term in the formula (2.7) depends on the time-parametrized path $\gamma(t)$ via the centrifugal potential energy V_μ and we refer to this as the *dynamic phase.* The second term in (2.7) depends only on the path and not on its parametrization and is referred to as the *geometric phase.* The factor $e \cdot \mathbf{J} e$ appearing in (2.7) is always a positive quantity and is simply the *locked inertia* of the body at the shape θ. Thus when $\mu = 0$, only the geometric phase remains and it corresponds to a *retrograde* rotation of the reference body 1.

Example (2 body problem)

$$\begin{aligned}\text{Let } \mu &= 0. \text{ Then,} \\ \triangle q_1 &= \triangle q_1^{\text{geometric}} \\ &= -\int_\gamma \frac{\tilde{I}_2 + \varepsilon\, d_1\, d_2\, cos\,(\theta)}{\tilde{I}_1 + \tilde{I}_2 + 2\,\varepsilon\, d_1\, d_2\, cos\,(\theta)}\; d\theta,\end{aligned} \tag{2.8}$$

where,

$$\begin{aligned}\tilde{I}_i &= I_i + \varepsilon\, d_i^2, \qquad i = 1,\, 2 \\ \varepsilon &= \frac{m_1\, m_2}{m_1 + m_2}\;,\end{aligned}$$

m_i and I_i are respectively the mass and inertia about the body-center-of -mass of the i^{th} body and d_i are as in Figure 2.

For interpretations of such phase shifts in the context of gymnastics and diving examples and for explicit integration of (2.8), see the paper of Frohlich [3]. Formulas analogous to (2.7) are known for the simple rigid body in 3 dimensions. See[9][6].

For $N > 2$, a closed curve γ can be the boundary of a smooth surface in shape space. Then, by Stokes' theorem, the geometric phase is given by,

$$\triangle q_1^{\text{geometric}} = -\oint\!\!\!\oint_\Gamma d\left(\frac{e \cdot \mathbf{J}\, M\, d\,\theta}{e \cdot \mathbf{J}\, e}\right) \tag{2.9}$$

where Γ is any surface in S with boundary $\partial\Gamma = \gamma$.

3. An Optimal Control Problem

Substituting (2.6) in (2.5), we get

$$\begin{aligned}\dot{q} &= \left[\frac{\mu}{e \cdot \mathbf{J}\, e}\right] e + \left[\mathbf{1} - \frac{e\, e^T\, \mathbf{J}}{e \cdot \mathbf{J}\, e}\right] M\, \dot{\theta}, \\ &= X_\mu(q) + D(q) v,\end{aligned} \tag{3.1}$$

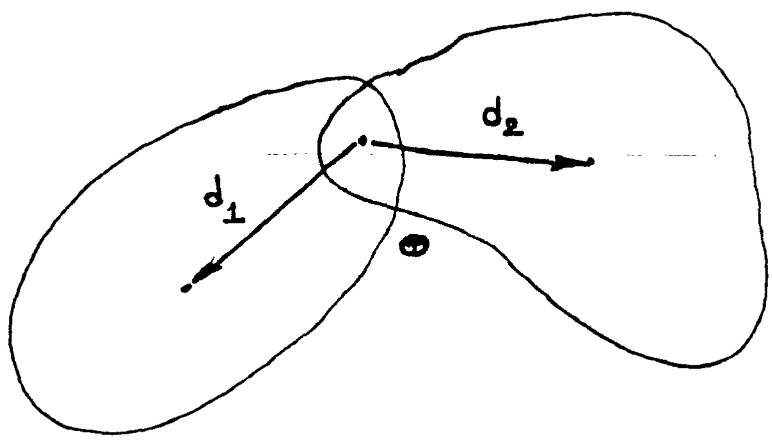

Figure 2

where $v = \dot{\theta}$, the shape velocity, is treated as a control. Immediately, we can ask:
(a) Is the system (3.1) accessible?
(b) Can we "solve" the variational problem,

$$\min_{v(\cdot)} \int_{t_0}^{t_1} < v, v >_S \; dt$$

subject to the conditions,

$$\begin{aligned} q(t_0) &= q^0 \\ q(t_1) &= q^1 \quad ? \end{aligned}$$

If q^0 and q^1 are in the same equivalence class in shape space, then, (b) is just the *isoholonomy problem* of Montgomery. Here $< \, , \, >_S$ is a riemannian metric on shape space.

If $\mu = 0$ and accessibility holds, then one derives a metric geometry on configuration space from the solution to (b). This is a singular/ subriemannian/ nonholonomic geometry in the sense of Brockett [2], Hermann [4], Strichartz [19], and Vershik - Gershkovich [20].

Letting $\mu = 0$, and using the flat metric on the shape space, the necessary conditions for an optimal control are that there exist a *state-costate pair,* $(q(t), \; p(t))$ satisfying

$$\begin{aligned} \dot{q} &= D(q) \; D^T(q)p \\ \dot{p} &= -\frac{\partial}{\partial q} \left(\frac{1}{2} \, p^T \; D(q) \; D^T(q)p \right) \\ v &= D^T(q)p \end{aligned} \tag{3.2}$$

See Sreenath [17] for a study of these equations for N small.

4. The Geometric Picture

Everything I have done above is part of a general geometric picture first recognized by Montgomery, Wilczek and Shapere. I give a simple treatment below.

Let $(Q,\ K,\ V,\ G)$ be a simple mechanical system with symmetry i.e. Q is the configuration space, K is the kinetic energy/riemannian metric, $V : Q \to \mathbb{R}$ is a G-invariant potential where G acts freely on Q by isometries. The system may be extended to include controls via the Lagrange - D'Alembert principle.

A basic object of interest is the principal bundle

$$\begin{array}{c} Q \\ \downarrow \\ Q/G = S \end{array}$$

where S is the shape space. Let $\mathcal{G}$ denote the Lie algebra of G and

$$\begin{aligned} \sigma_q : \mathcal{G} &\to TQ_q \\ \xi &\mapsto \xi_Q(q) \end{aligned}$$

define the infinitesimal action. Let $\mathbb{I}_q : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ define a symmetric bilinear form, $\mathbb{I}_q(\xi,\ \eta) = K(\sigma_q \xi,\ \sigma_q \eta)$ for all $\xi, \eta \in \mathcal{G}$. We also define $\mathbb{I}_q^\flat : \mathcal{G} \to \mathcal{G}^*$ by setting $\mathbb{I}_q^\flat = \sigma_q^* K^\flat \sigma_q$ and $K^\flat : TQ_q \to T^*Q_q$ is the usual Legendre transform. It is an easy exercise to verify that

$$\begin{aligned} J^\# : TQ &\to \mathcal{G}^* \\ w_q &\mapsto J^\#(w_q) \end{aligned}$$

defined by setting $J^\#(w_q)\xi = (K^\flat w_q)(\xi_Q(q))$ is the conserved *momentum map* of the *free* hamiltonian system. *Admissible controls are internal controls, i.e. they also leave $J^\#$ invariant.* Consider the splitting,

$$\begin{aligned} TQ_q &= (Vert)_q \oplus (Hor)_q \\ w_q &= \sigma_q\ \mathbb{I}_q^{\flat^{-1}} \mu + (w_q - \sigma_q\ \mathbb{I}_q^{\flat^{-1}} \mu) \end{aligned}$$

where $\mu = J^\#(w_q)$ · The splitting has the equivariance property w.r.t. the G action and defines a principal connection. It appears that this principal connection was known to Smale and Kummer. One can define a $\mathcal{G}$-valued 1-form

$$\begin{aligned} A : Q &\to T^*Q \otimes \mathcal{G} \\ A_q : TQ_q &\mapsto \mathcal{G} \\ w &\mapsto \mathbb{I}_q^{\flat^{-1}} M(q,\ w) \\ &= (\sigma_q^* K^\flat \sigma_q)^{-1} \sigma_q^* K^\flat w. \end{aligned}$$

The holonomy of this connection captures the geometric phase. In the example of this paper the splitting (3.1) defines a connection. The Lie algebra = $\mathbb{R}$ and hence the connection form is just an $\mathbb{R}$-valued form. Note that if $f : Q/G \rightarrow Q$ is a cross section of the bundle $Q \rightarrow Q/G$ then one can *pull-down* A to $\tilde{A} = f^*A : Q/G \rightarrow T^*Q \otimes \mathcal{G}$. In our example, since the bundle over shape space is trivial, one has cross-sections and the pull-down version of the connection 1-form is precisely given by the form

$$-\frac{e \cdot \mathbf{J}\, M\, d\theta}{e \cdot \mathbf{J}\, e}$$

in equation (2.7).

Wilczek and Shapere refer to $\tilde{A}$ as the master gauge field [14], [13].

5. Final Remarks

In a complete manuscript under preparation we discuss other aspects of the problem of this paper including the accessibility question (recall Ambrose - Singer theorem) in geometric terms.

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